ON POINTED HOPF ALGEBRAS ASSOCIATED TO UNMIXED CONJUGACY CLASSES IN $\mathbb{S}_n$

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Abstract. Let $\pi \in \mathbb{S}_n$ be a product of disjoint cycles of the same length, $\mathcal{C}$ the conjugacy class of π and ρ an irreducible representation of the isotropy group of π. We prove that either the Nichols algebra $\mathfrak{B}(\mathcal{C}, \rho)$ is infinite-dimensional, or the braiding of the Yetter-Drinfeld module is negative.

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Introduction

Hopf algebras have important applications in mathematics and mathematical physics. Indeed, Hopf algebras give rise to finite tensor categories in the sense of [ENO, EO] through their categories of representations. In this way, for instance, semisimple Hopf algebras are present in a fundamental way in rational conformal field theories. Also, non-semisimple Hopf algebras are related to logarithmic conformal field theories [Ga]. It is natural to expect that classification results on finite-dimensional Hopf algebras would

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have a significant impact in those areas. Needless to say, classification efforts often come out in discovery of new examples.

This article, in the line of [AZ], is a contribution to the classification of finite-dimensional complex pointed Hopf algebras H with $G(H)$ non-abelian, in the framework of the Lifting Method [AS2]. Let G be a finite group. An important stage in the proposed approach to classification of finite-dimensional complex pointed Hopf algebras H with $G(H) = G$ is the determination of all Yetter-Drinfeld modules V over the group algebra of G such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional. When the finite group G is abelian, this amounts to the study of Nichols algebras of diagonal type. In this context, substantial results were reached in [H4], see also [AS1, H1, H2, H3].

For a general finite group G, irreducible Yetter-Drinfeld modules (up to isomorphisms) over $\mathbb{C}G$ are parametrized by pairs $(\mathcal{C}, \rho)$ where $\mathcal{C}$ is a conjugacy class of G and ρ is an irreducible representation of the centralizer G^s of a fixed $s \in \mathcal{C}$. Say $M(\mathcal{C}, \rho)$ is the irreducible Yetter-Drinfeld module corresponding to $(\mathcal{C}, \rho)$ and $\mathfrak{B}(\mathcal{C}, \rho)$ is its Nichols algebra. As in [AZ], we use the following strategy.

Strategy. *Given $(\mathcal{C}, \rho)$, find a braided subspace W of $M(\mathcal{C}, \rho)$ of diagonal type. Check if the dimension of the Nichols algebra $\mathfrak{B}(W)$ is infinite using the above mentioned results. If so, then necessarily $\dim(\mathfrak{B}(\mathcal{C}, \rho)) = \infty$.*

Note that this strategy only involves the determination of a braided subspace W of $M(\mathcal{C}, \rho)$ of diagonal type. This is an elementary problem of group theory and its solution does not requires any knowledge of pointed Hopf algebras or Nichols algebras.

In this paper, we deal with the groups $G = \mathbb{S}_n$. The orbit of $\pi \in \mathbb{S}_n$ is determined by the lengths of the disjoint cycles in the decomposition of π. We call π *unmixed* if all those lengths are equal, and *mixed* otherwise.

The main results in this paper are summarized in the following statement, see below for the unexplained notations.

Theorem 1. *Let $\pi \in \mathbb{S}_{kn}$ be unmixed, say of type (k^n), and $\rho \in \widehat{\mathbb{S}_{kn}^{\pi}}$.*

(A) If k is odd, then $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$.

(B) Assume that $k = 2$.

- (i) *If n is even, then $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$.*
- (ii) *Assume that n is odd, $n > 1$. If $\rho = \chi_{(n)} \otimes \epsilon$ or $\chi_{(n)} \otimes \operatorname{sgn}$, then the braiding is negative. Otherwise, $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$.*

(C) Assume that $k = 2r$, with $r > 1$.

- (i) *If $n = 1$, then $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$ if $\rho = \chi_\omega$, with $\omega \neq -1$, and the braiding is negative if $\rho = \chi_\omega$, with $\omega = -1$.*
- (ii) *Assume that $n > 1$. If $\deg \rho > 1$, or if $\deg \rho = 1$ and $\rho(\pi) \neq -1$, then $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$. Assume that $\deg \rho = 1$ and $\rho(\pi) = -1$. If*

$\rho = \chi_{r,\dots,r} \otimes \mu$, with r even or odd, or if $\rho = \chi_{c,\dots,c} \otimes \mu$, with r even and $c = \frac{r}{2}$ or $\frac{3r}{2}$, then the braiding is negative, where $\mu = \epsilon$ or sgn*; otherwise,* $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$.

Part (A) follows from [AZ], see Lemma 2.1; [AZ, Theor. 2.7], Propositions 2.4, 2.11 and Theorem 2.12 for part (B); and Proposition 2.2, Theorem 2.14, Proposition 2.15 and Theorem 2.16 for part (C). Briefly, for the braided vector spaces considered in this paper, either the Nichols algebra is infinite-dimensional or the braiding is negative. The notation in the Theorem can be found in the pages 5, 6, 6 and 21 for negative braiding, $\chi_{(n)}$, χ_ω and $\chi_{r,\dots,r}$, respectively.

We shall consider the mixed case in a subsequent publication; partial results follow from the unmixed case and [AZ, Prop. 2.6].

1. Preliminaries

Our main reference for the classification problem of pointed Hopf algebras is [AS2]. We denote by $\widehat{G}$ the set of isomorphism classes of irreducible representations of a finite group G. Consequently, the group of characters of a finite abelian group Γ is denoted $\widehat{\Gamma}$. We shall often denote a representant of a class in $\widehat{G}$ with the same symbol as the class itself. If $\rho \in \widehat{G}$, $\deg \rho$ is the dimension of the vector space V affording ρ. If V is a Γ-module then V^χ is the isotypic component of type $\chi \in \widehat{\Gamma}$. We shall use the rack notation $g \triangleright h = ghg^{-1}$, $g, h \in G$. A left comodule over the group algebra $\mathbb{C}G$ is the same as a G-graded vector space: if M is a $\mathbb{C}G$-comodule, then $M = \oplus_{h\in G} M_h$ where $M_h = \{m \in M : \delta(m) = h \otimes m\}$.

We denote by $\mathbb{G}_n$ the group of n-th roots of 1 in $\mathbb{C}$.

1.1. **Yetter-Drinfeld modules over a finite group.** Let G be a finite group. A Yetter-Drinfeld module over G is a left G-module and left $\mathbb{C}G$-comodule M satisfying the compatibility condition $\delta(g.m) = ghg^{-1} \otimes g.m$, for all $m \in M_h$, $g, h \in G$. It is well-known that Yetter-Drinfeld modules over G are completely reducible, and that irreducible Yetter-Drinfeld modules over G are parameterized by pairs $(\mathcal{C}, \rho)$ where $\mathcal{C}$ is a conjugacy class in G and ρ is an irreducible representation of the isotropy subgroup G^s of a fixed point $s \in \mathcal{C}$ on a vector space V. We denote the corresponding Yetter-Drinfeld module by $M(\mathcal{C}, \rho)$; a precise description is as follows. Let $t_1 = s$, $\dots$, t_M be a numeration of $\mathcal{C}$ and let $g_i \in G$ such that $g_i \triangleright s = t_i$ for all $1 \le i \le M$. Then $M(\mathcal{C}, \rho) = \oplus_{1\le i\le M}\, g_i \otimes V$. Let $g_i v := g_i \otimes v \in M(\mathcal{C}, \rho)$, $1 \le i \le M$, $v \in V$. If $v \in V$ and $1 \le i \le M$, then the action of $g \in G$ is given by $g \cdot (g_i v) = g_j(\gamma \cdot v)$, where $gg_i = g_j\gamma$, for some $1 \le j \le M$ and $\gamma \in G^s$, and the coaction is given by $\delta(g_i v) = t_i \otimes g_i v$. The Yetter-Drinfeld module $M(\mathcal{C}, \rho)$ is a braided vector space (see below) with braiding

$$c(g_i v \otimes g_j w) = t_i \cdot (g_j w) \otimes g_i v = g_h(\gamma \cdot w) \otimes g_i v, \tag{1}$$

for any $1 \leq i, j \leq M$, $v, w \in V$, where $t_i g_j = g_h \gamma$ for unique h, $1 \leq h \leq M$ and $\gamma \in G^s$. Since $s \in Z(G^s)$, the Schur Lemma implies that

$$s \text{ acts by a scalar } q_{ss} \text{ on } V. \tag{2}$$

Notice that $M(\mathcal{C}, \rho)$ depends on the element s in $\mathcal{C}$ and ρ in $\widehat{G^s}$. Let $s, \tilde{s} \in \mathcal{C}$ and let $g \in G$ such that $gsg^{-1} = \tilde{s}$; thus $gG^s g^{-1} = G^{\tilde{s}}$. Let $\widetilde{\rho} \in \widehat{G^{\tilde{s}}}$ the pullback of $\rho \in \widehat{G^s}$ via conjugate by g. Then $M(\mathcal{C}, \rho) = M(\mathcal{C}, \widetilde{\rho})$; in particular

$$\text{the images of } \rho \text{ and } \widetilde{\rho} \text{ in } GL(V) \text{ coincide.} \tag{3}$$

1.2. **On Nichols algebras.** Let (V, c) be a braided vector space, i. e. V is a vector space and $c : V \otimes V \to V \otimes V$ is an automorphism satisfying the braid equation $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$. Then $\mathfrak{B}(V)$ denotes the Nichols algebra of V, see [AS2]. The Nichols algebra of a Yetter-Drinfeld module $M(\mathcal{C}, \rho)$ will be denoted just by $\mathfrak{B}(\mathcal{C}, \rho)$.

Lemma 1.1. *If W is a subspace of V such that $c(W \otimes W) = W \otimes W$ and $\dim \mathfrak{B}(W) = \infty$ then $\dim \mathfrak{B}(V) = \infty$.* □

Indeed, $\mathfrak{B}(W) \subset \mathfrak{B}(V)$. A first application of the Lemma is when there exists $v \in V - 0$ such that $c(v \otimes v) = v \otimes v$; then $\dim \mathfrak{B}(V) = \infty$. In particular, if $V = M(\mathcal{C}, \rho)$ and $q_{ss} = 1$, see (2), then $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$.

A braided vector space (V, c) is of *diagonal type* if there exists a basis $v_1, \dots, v_\theta$ of V and non-zero scalars q_{ij}, $1 \leq i, j \leq \theta$, such that $c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$, for all $1 \leq i, j \leq \theta$. A braided vector space (V, c) of diagonal type is of *Cartan type* if $q_{ii} \neq 1$ is a root of 1 for all i: $1 \leq i \leq \theta$, and there exists $a_{ij} \in \mathbb{Z}$, $-\operatorname{ord} q_{ii} < a_{ij} \leq 0$ such that $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ for all $1 \leq i \neq j \leq \theta$. Set $a_{ii} = 2$ for al $1 \leq i \leq \theta$. Then $(a_{ij})_{1 \leq i, j \leq \theta}$ is a generalized Cartan matrix.

Theorem 1.2. *(*[H3, Th. 4]*, see also* [AS1, Th. 1]*). Let (V, c) be a braided vector space of Cartan type. Then $\dim \mathfrak{B}(V) < \infty$ if and only if the Cartan matrix is of finite type.* □

1.3. **Abelian subspaces of a braided vector space.** Our aim now is to describe a recipe for finding braided subspaces W of diagonal type of a braided vector space $M(\mathcal{C}, \rho)$.

Let $(X, \triangleright)$ be a rack. Let $q : X \times X \to \mathbb{C}^\times$ be a rack 2-cocycle and let $(\mathbb{C}X, c_q)$ be the associated braided vector space, that is $\mathbb{C}X$ is a vector space with a basis e_x, $x \in X$, and $c_q(e_x \otimes e_y) = q_{x,y} e_{x \triangleright y} \otimes e_x$, for all $x, y \in X$. Let us say that a subrack T of X is *abelian* if $i \triangleright j = j$ for all $i, j \in T$. If T is an abelian subrack of X then $\mathbb{C}T$ is a braided vector subspace of $(\mathbb{C}X, c_q)$.

Definition 1.3. We say that $(\mathbb{C}X, c_q)$ is *weakly finite* if $\dim \mathfrak{B}(\mathbb{C}T) < \infty$ for any abelian subrack T of X.

Our interest is to check when $(\mathbb{C}X, c_q)$ is not weakly finite, for then $\dim \mathfrak{B}(\mathbb{C}X) = \infty$.

We shall say that $(\mathbb{C}X, c_q)$ is *negative* if $q_{ii} = -1$ and $q_{ij}q_{ji} = 1$ for all $i, j \in T$ (hence $\mathfrak{B}(\mathbb{C}T)$ is an exterior algebra and $\dim \mathfrak{B}(\mathbb{C}T) = 2^{\operatorname{card} T}$) and for any abelian subrack T of X. This is a very particular case, but we note that almost all known braided vector spaces with finite dimensional Nichols algebra that "do not come from the abelian case" are negative. See [Gñ].

Let now G be a finite group, $\mathcal{C}$ a conjugacy class in G, $\rho \in \widehat{G^s}$ with $s \in \mathcal{C}$ fixed. As in subsection 1.1, we fix a numeration $t_1 = s, \dots, t_M$ of $\mathcal{C}$ and $g_i \in G$ such that $g_i \triangleright s = t_i$ for all $1 \le i \le M$. Let $T = \{t_i : i \in I\}$ be an abelian subrack of $\mathcal{C}$, $I \subset \{1, \dots, M\}$. Let $i, j \in I$. Then the following are equivalent:

(i) $t_i \triangleright t_j = t_j$, that is, t_i and t_j commute.
(ii) $\gamma_{ij} := g_j^{-1} t_i g_j \in G^s$.

Let V be the vector space affording ρ. For every $v, w \in V$, we have that

$$c(g_i v \otimes g_j w) = g_j(\gamma_{ij} \cdot w) \otimes g_i v \tag{4}$$

Let $v_1, \dots, v_R$ be simultaneous eigenvectors of γ_{ij}, $i, j \in I$. Then

$$W = \mathbb{C} - \text{span of } g_i v_j, \quad i \in I,\ 1 \le j \le R,$$

is a braided subspace of diagonal type of dimension $\operatorname{card} T R$. Note that R depends not only on T but also on the representation ρ; for instance if ρ is a character then $R = 1 = \dim V$, and $M(\mathcal{C}, \rho)$ is of rack type.

Notice that the action of G on $\mathcal{C}$ (by conjugation) preserves abelian racks. It is then natural to ask: Are two *maximal* abelian subracks of $\mathcal{C}$ conjugated by some $g \in G$?

2. On Nichols algebras over $\mathbb{S}_n$

2.1. **Notation on the groups $\mathbb{S}_n$.** Assume that in the decomposition of $\pi \in \mathbb{S}_n$ as product of disjoint cycles, there are m_j cycles of length j, $1 \le j \le n$. Then the type of $\pi \in \mathbb{S}_n$ is the symbol $(1^{m_1}, 2^{m_2}, \dots, n^{m_n})$; we may omit j^{m_j} when $m_j = 0$. The conjugacy class $\mathcal{O}_\pi$ of π is the set of all permutations in $\mathbb{S}_n$ with the same type as π; we may use the type as a subscript of a conjugacy class as well. We say that π is *unmixed* if the type of π is (k^n), i.e. if π is a product of disjoint cycles of the same length. Let us assume that π is unmixed. It is known that the isotropy subgroup of π satisfies $\mathbb{S}_{kn}^\pi \simeq \Gamma \rtimes \mathbb{S}_n$, where $\Gamma \simeq (\mathbb{Z}/k)^n$ is generated by the k-cycles in π and $\mathbb{S}_n$ permutes these cycles. This leads us to the representation theory of groups of the form $\Gamma \rtimes A$, with Γ abelian. The irreducible representations of $\Gamma \rtimes A$ are described as follows. Let $\chi \in \widehat{\Gamma}$ and let A^χ be the isotropy group with respect to the natural action of A on $\widehat{\Gamma}$. Let $\mu \in \widehat{A^\chi}$ and let ρ be the induced representation $\rho = \operatorname{Ind}_{\Gamma \rtimes A^\chi}^{\Gamma \rtimes A}(\chi \otimes \mu)$. Then ρ is irreducible and any irreducible representation of $\Gamma \rtimes A$ is isomorphic to one of this form, for unique χ up to the action of A and $\mu \in \widehat{A^\chi}$, see [S, 8.2].

Let ϵ and sgn denote the trivial and sign representation of $\mathbb{S}_n$, respectively.

2.2. **Nichols algebras corresponding to even cycles.** The next application of Theorem 1.2 is [AZ, Lemma 2.3].

Lemma 2.1. *If* $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) < \infty$ *then* $q_{\pi\pi} = -1$ *and* π *has even order.* $\square$

In this subsection we consider the case when the type of π in $\mathbb{S}_k$ is (k), with $2 < k$ even. Thus, $\mathcal{O}_\pi$ is the set of k-cycles in $\mathbb{S}_k$. Fix $\pi = (12\dots k)$; the isotropy subgroup is $\mathbb{S}_k^\pi = \langle\pi\rangle \simeq \mathbb{Z}_k$. If (j,k) denotes the highest common divisor of j and k, then the maximal abelian subrack of $\mathcal{O}_\pi$ is

$$T = \{\pi^j : (j,k) = 1\}.$$

Clearly, $\operatorname{card} T \geq 2$. Let $\omega \in \mathbb{G}_k$, let χ_ω be the character of $\mathbb{S}_k^\pi$ defined by $\chi_\omega(\pi) = \omega$, let $M(\mathcal{O}_\pi, \chi_\omega)$ be the corresponding Yetter-Drinfeld module and let $\mathfrak{B}(\mathcal{O}_\pi, \chi_\omega)$ be its Nichols algebra. We conclude from Lemma 2.1:

Proposition 2.2. *Let* $\pi \in \mathbb{S}_k$ *of type* (k)*,* k *even. Let* $\omega \in \mathbb{G}_k$*. Then* $\dim \mathfrak{B}(\mathcal{O}_\pi, \chi_\omega) = \infty$ *if* $\omega \neq -1$*, and the braiding is negative if* $\omega = -1$*.* $\square$

2.3. **Nichols algebras of orbits with n transpositions.** In this subsection we consider the case when the type of π in $\mathbb{S}_{2n}$ is (2^n), $n > 1$. Thus, $\mathcal{O}_\pi$ is the set of permutations in $\mathbb{S}_{2n}$ that are the product of n disjoint transpositions. Fix $\pi = A_1 \dots A_n$ in $\mathbb{S}_{2n}$, with $A_i = (2i-1\ 2i)$. The isotropy subgroup is

$$\mathbb{S}_{2n}^\pi = \langle A_1, \dots, A_n\rangle \rtimes \langle B_1, \dots, B_{n-1}\rangle \simeq \mathbb{Z}_2^n \rtimes \mathbb{S}_n,$$

where $B_j = (2j-1 \quad 2j+1)\,(2j \quad 2j+2)$, $1 \leq j \leq n-1$; the relations are

$$\begin{aligned} &A_i^2 = \mathrm{id} = B_j^2, && A_iA_j = A_jA_i, && A_iB_j = B_jA_i\,,\ i \neq j, j+1,\\ &A_jB_j = B_jA_{j+1}, && B_iB_j = B_jB_i\,,\ |i-j| > 1, && B_jB_{j\pm1}B_j = B_{j\pm1}B_jB_{j\pm1}, \end{aligned}$$

2.3.1. *Irreducible representations of* $\mathbb{Z}_2^n \rtimes \mathbb{S}_n$. We first list the irreducible representations of $\mathbb{Z}_2^n \rtimes \mathbb{S}_n$. Let $e_i \in \mathbb{Z}_2^n$ be the element with 1 in the i-th component and 0 elsewhere; let $\chi_i \in \widehat{\mathbb{Z}_2^n}$ be given by $\chi_i(e_j) = (-1)^{\delta_{i,j}}$. The irreducible representations of $\mathbb{Z}_2^n$ are the linear characters

$$\chi_{i_1,\dots,i_k} := \chi_{i_1} \dots \chi_{i_k}, \quad 0 \leq k \leq n, \quad 1 \leq i_1 < \dots < i_k \leq n,$$

where $k = 0$ corresponds to the trivial representation $\chi_{(0)}$ of $\mathbb{Z}_2^n$. Let $\chi_{(k)} := \chi_{1,\dots,k}$. The action of $\mathbb{S}_n$ on $\mathbb{Z}_2^n$ induces a natural action of $\mathbb{S}_n$ on $\widehat{\mathbb{Z}_2^n}$; the orbit and the isotropy subgroup of $\chi = \chi_{i_1,\dots,i_k} \in \widehat{\mathbb{Z}_2^n}$ are

$$\mathcal{O}_\chi = \{\chi_{j_1,\dots,j_k} : 1 \leq j_1 < \dots < j_k \leq n\}, \qquad \mathbb{S}_n^\chi \simeq \mathbb{S}_{n-k} \times \mathbb{S}_k.$$

Thus the characters $\chi_{(k)}$, $0 \le k \le n$, are a complete set of representatives of the orbits in $\widehat{\mathbb{Z}_2^n}$. As discussed in subsection 2.1, all the irreducible representations of $\mathbb{Z}_2^n \rtimes \mathbb{S}_n$ are of the form

$$\rho = \rho_{\chi_{(k)},\mu} = \operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(k)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n} (\chi_{(k)} \otimes \mu), \qquad 0 \le k \le n, \quad \mu \in \widehat{\mathbb{S}_n^{\chi_{(k)}}}.$$

There are $\sum_{k=0}^n \mathcal{P}(n-k)\mathcal{P}(k)$ irreducible representations of $\mathbb{Z}_2^n \rtimes \mathbb{S}_n$, where $\mathcal{P}$ is the partition function, but we do not need to consider all of them.

Remark 2.3. If k is even, then $\rho_{\chi_{(k)},\mu}(\pi)$ acts by $q_{\pi\pi} = 1$, for any $\mu \in \widehat{\mathbb{S}_n^{\chi_{(k)}}}$. Thus $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, by Lemma 2.1. So, we can assume k odd.

Let $t_1 = \pi, \dots, t_M$ be a numeration of $\mathcal{O}_\pi$, as in subsection 1.1; we can assume that the elements $g_1, \dots, g_M$ satisfying $g_i \triangleright \pi = t_i$, are involutions.

Proposition 2.4. *If n is odd, then the braided vector space associated to $\chi_{(n)} \otimes \epsilon$ or to $\chi_{(n)} \otimes \operatorname{sgn}$ is negative.*

Proof. Assume that $t_i \neq t_j$ commute. We must show that $q_{ii} = -1$, $q_{jj} = -1$ and $q_{ij}q_{ji} = 1$. The first two conditions are fulfilled because $t_l g_l = g_l \pi$, $1 \le l \le M$. For the third, note that $\gamma_{ij} := g_j^{-1} t_i g_j$ and $\gamma_{ji} := g_i^{-1} t_j g_i$ are in $\mathbb{S}_{2n}^\pi$, see subsection 1.3; so we can write

$$\gamma_{ij} = A_1^{d_1} \cdots A_n^{d_n} B_{h_1} \cdots B_{h_P}, \qquad \gamma_{ji} = A_1^{e_1} \cdots A_n^{e_n} B_{l_1} \cdots B_{l_Q},$$

where $d_1, \dots, d_n, e_1, \dots, e_n \in \{0, 1\}$. Since γ_{ij}, $\gamma_{ji} \in \mathcal{O}_\pi$, signs of the permutations γ_{ij} and γ_{ji} are equal to the sign of π, which is -1, because n is odd and the sign of every permutation B_l is 1, $1 \le l \le n-1$. This implies that $d_1 + \cdots + d_n$ and $e_1 + \cdots + e_n$ are odd. Now, since $t_i g_j = g_j \gamma_{ij}$ and $t_j g_i = g_i \gamma_{ji}$ then $q_{ij} q_{ji} = \rho(\gamma_{ij}\gamma_{ji})$. We consider the two cases.

(a) *Assume that* $\rho = \chi_{(n)} \otimes \epsilon$. In this case, the result follows because

$$\rho(\gamma_{ij}\gamma_{ji}) = (-1)^{d_1+\cdots+d_n}(-1)^{e_1+\cdots+e_n} = 1.$$

(b) *Assume that* $\rho = \chi_{(n)} \otimes \operatorname{sgn}$. If $t_i = \pi$, then $\gamma_{ij} = \gamma_{ji} = t_j$, because g_j is an involution, and the result follows.

We will see that the general case follows from the case $t_i = \pi$. By definition, $M(\mathcal{C}, \rho)$ is a $\mathbb{S}_{2n}$-comodule, with coaction given by $\delta(g_l v) = t_l \otimes g_l v$, where $V = \mathbb{C}$ - span of v. Then $M(\mathcal{C}, \rho) = \oplus_{\tau \in \mathbb{S}_{2n}} M(\mathcal{C}, \rho)_\tau$, where

$$M(\mathcal{C}, \rho)_\tau := \{m \in M(\mathcal{C}, \rho) \,:\, \delta(m) = \tau \otimes m\}.$$

Clearly, $M(\mathcal{C}, \rho)_\tau = g_i V$, if $\tau = t_i$, and $M(\mathcal{C}, \rho)_\tau = 0$, if $\tau \notin \mathcal{O}_\pi$.

Let us call $\widetilde{t_1} := t_i$; then $M(\mathcal{C}, \rho) \simeq \operatorname{Ind}_{\mathbb{S}_{2n}^{\widetilde{t_1}}}^{\mathbb{S}_{2n}} \widetilde{V}$, where $\widetilde{V}$ is an irreducible representation of dimension 1 of $\mathbb{S}_{2n}^{\widetilde{t_1}} \simeq \mathbb{S}_{2n}^\pi$, see (3). Let $\widetilde{t_2} := t_j$, $\widetilde{g_1} := \mathrm{id}$

and let $\widetilde{g_2}$ be such that $\widetilde{g_2}\widetilde{t_1}\widetilde{g_2}^{-1} = \widetilde{t_2}$. Thus, there exists $\widetilde{v}$ in $\widetilde{V}$ which satisfies

$$g_i\mathbb{C}v = M(\mathcal{C},\rho)_{t_i} = \widetilde{g_1}\mathbb{C}\widetilde{v}, \qquad g_j\mathbb{C}v = M(\mathcal{C},\rho)_{t_j} = \widetilde{g_2}\mathbb{C}\widetilde{v}.$$

Let us say $\widetilde{g_1}\widetilde{v} = \lambda_1 g_i v$ and $\widetilde{g_2}\widetilde{v} = \lambda_2 g_j v$. Then

$$\begin{aligned} c(\widetilde{g_1}\widetilde{v} \otimes \widetilde{g_2}\widetilde{v}) &= c(\lambda_1 g_i v \otimes \lambda_2 g_j v) = \lambda_1\lambda_2 c(g_i v \otimes g_j v) \\ &= \lambda_1\lambda_2 q_{ij} g_j v \otimes g_i v = q_{ij}\widetilde{g_2}\widetilde{v} \otimes \widetilde{g_1}\widetilde{v}, \end{aligned}$$

and on the other hand, $c(\widetilde{g_1}\widetilde{v}\otimes\widetilde{g_2}\widetilde{v}) = \widetilde{q_{12}}\widetilde{g_l}\widetilde{v}\otimes\widetilde{g_1}\widetilde{v}$; so, $q_{ij} = \widetilde{q_{12}}$. Analogously, $q_{ji} = \widetilde{q_{21}}$. Hence, $q_{ij}q_{ji} = \widetilde{q_{12}}\widetilde{q_{21}} = 1$, and the result follows. □

We proceed now to consider the different cases according to the parity of n. The case $n = 2$ is contained in [AZ]. We first consider $n = 3$, 4 and then the general cases n even and n odd.

2.3.2. *Case* $n = 3$. Let $\pi = (1\,2)(3\,4)(5\,6)$ in $\mathbb{S}_6$. Then $\mathcal{O}_\pi$ has 15 elements and the isotropy subgroup of π is

$$\begin{aligned} \mathbb{S}_6^\pi &= \langle A_1 = (1\,2), A_2 = (3\,4), A_3 = (5\,6)\rangle \rtimes \langle B = (1\,3)(2\,4), C = (1\,3\,5)(2\,4\,6)\rangle \\ &\simeq \mathbb{Z}_2^3 \rtimes \mathbb{S}_3. \end{aligned}$$

The defining relations for the generators A_1, A_2, A_3, B and C are $B^2 = C^3 = 1 = A_i^2$, $A_iA_j = A_jA_i$, $BCB = C^{-1}$ and

$$\begin{aligned} BA_1B &= A_2, & BA_2B &= A_1, & BA_3B &= A_3, \\ CA_1C^{-1} &= A_2, & CA_2C^{-1} &= A_3, & CA_3C^{-1} &= A_1. \end{aligned}$$

By section 2.3.1, the irreducible representations of $\mathbb{S}_6^\pi$ are:
(1) Four characters $\chi_{\pm,\pm}$ given by $\chi_{\pm,\pm}(A_i) = \pm 1$ (the first subindex), $\chi_{\pm,\pm}(B) = \pm 1$ (the second subindex), $\chi_{\pm,\pm}(C) = 1$.
(2) Two 2-dimensional representations $\theta_\pm$ given by

$$\theta_\pm(A_i) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \theta_\pm(B) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \theta_\pm(C) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix},$$

where $\omega \in \mathbb{C}_3^\times$ is a primitive 3-th root of 1.
(3) Four 3-dimensional representations $\phi_\pm$, $\psi_\pm$ given by

$$\phi_\pm(A_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \phi_\pm(A_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\phi_\pm(A_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \phi_\pm(B) = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{pmatrix}, \phi_\pm(C) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$\psi_\pm(A_1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \psi_\pm(A_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\psi_{\pm}(A_3)=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\psi_{\pm}(B)=\begin{pmatrix}\pm1&0&0\\0&0&\pm1\\0&\pm1&0\end{pmatrix},\psi_{\pm}(C)=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}.$$

The representations $\chi_{-,\pm}$ are the $\chi_{(n)}\otimes\epsilon$ and $\chi_{(n)}\otimes\mathrm{sgn}$ in Proposition 2.4, thus we can not decide the dimension of their Nichols algebras. For the others, we have:

Proposition 2.5. *Let* $\pi\in\mathbb{S}_6$ *with type* (2^3). *Let* ρ *be in* $\widehat{\mathbb{S}_6^\pi}$. *If* $\rho\neq\chi_{-,\pm}$ *then* $\dim\mathfrak{B}(\mathcal{O}_\pi,\rho)=\infty$.

Proof. We can assume that $\pi=A_1A_2A_3=(12)(34)(56)$. Let $\rho\in\widehat{\mathbb{S}_n^\pi}$ and V the vector space affording ρ. We look for a braided subspace of diagonal type of $M(\mathcal{O}_{2,2,2},\rho)$. Set $\alpha=(12)(35)(46)$, $\beta=(12)(36)(45)$ in $\mathcal{O}_{2,2,2}$; if $\sigma_1=\mathrm{id}$, $\sigma_2=(4\,5)$, $\sigma_3=(4\,6)$ then

$$\sigma_1\triangleright\pi=\pi,\quad\sigma_2\triangleright\pi=\alpha,\quad\sigma_3\triangleright\pi=\beta.$$

Let $\sigma_jv:=\sigma_j\otimes v$, $v\in V$, $1\leq j\leq 3$. The coaction is given by $\delta(\sigma_jv)=\sigma_j\triangleright\pi\otimes\sigma_jv$; we need the action of the elements π, α, β, which is

$$\begin{aligned}&\pi\cdot\sigma_1v=\sigma_1\rho(\pi)(v), &&\pi\cdot\sigma_2v=\sigma_2\rho(\alpha)(v), &&\pi\cdot\sigma_3v=\sigma_3\rho(\beta)(v),\\ &\alpha\cdot\sigma_1v=\sigma_1\rho(\alpha)(v), &&\alpha\cdot\sigma_2v=\sigma_2\rho(\pi)(v), &&\alpha\cdot\sigma_3v=\sigma_3\rho(\alpha)(v),\\ &\beta\cdot\sigma_1v=\sigma_1\rho(\beta)(v), &&\beta\cdot\sigma_2v=\sigma_2\rho(\beta)(v), &&\beta\cdot\sigma_3v=\sigma_3\rho(\pi)(v).\end{aligned}$$

Hence the braiding is given, by

$$\begin{aligned}&c(\sigma_1v\otimes\sigma_2w)=\sigma_2\rho(\alpha)(w)\otimes\sigma_1v, &&c(\sigma_1v\otimes\sigma_3w)=\sigma_3\rho(\beta)(w)\otimes\sigma_1v,\\ &c(\sigma_2v\otimes\sigma_1w)=\sigma_1\rho(\alpha)(w)\otimes\sigma_2v, &&c(\sigma_2v\otimes\sigma_3w)=\sigma_3\rho(\alpha)(w)\otimes\sigma_2v,\\ &c(\sigma_3v\otimes\sigma_1w)=\sigma_1\rho(\beta)(w)\otimes\sigma_3v, &&c(\sigma_3v\otimes\sigma_2w)=\sigma_2\rho(\beta)(w)\otimes\sigma_3v,\end{aligned}$$

and $c(\sigma_jv\otimes\sigma_jw)=(\sigma_j\triangleright\pi)\cdot\sigma_jw\otimes\sigma_jv=\sigma_j\rho(\pi)(w)\otimes\sigma_jv=-\sigma_jw\otimes\sigma_jv$, for all $1\leq j\leq 3$ and $v,w\in V$.

We now consider the different possibilities for ρ. If $\rho=\chi_{+,\pm}$, θ_+ or $\psi_\pm$ then $q_{\pi\pi}=1$ and $\dim\mathfrak{B}(\mathcal{O}_{2,2,2},\rho)=\infty$, by Lemma 1.1.

If $\rho=\theta_-$ then $\rho(\alpha)=\rho(\beta)=\begin{pmatrix}0&-\omega^{-1}\\-\omega&0\end{pmatrix}$. Choose $v_1=\begin{pmatrix}1\\-\omega\end{pmatrix}$ and $v_2=\begin{pmatrix}1\\\omega\end{pmatrix}$. Hence $\rho(\alpha)(v_1)=\rho(\beta)(v_1)=v_1$, $\rho(\alpha)(v_2)=\rho(\beta)(v_2)=-v_2$. Therefore the braiding is diagonal of Cartan type in the basis

$$w_1=\sigma_1v_1,\ w_2=\sigma_1v_2,\ w_3=\sigma_2v_1,\ w_4=\sigma_2v_2,\ w_5=\sigma_3v_1,\ w_6=\sigma_3v_2.$$

The corresponding Dynkin diagram is $A_5^{(1)}$, which is affine. By Theorem 1.2, $\dim\mathfrak{B}(\mathcal{O}_{2,2,2},\theta_-)=\infty$.

Assume now that $\rho = \phi_+$. Then $\phi_+(\alpha) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,
$\phi_+(\beta) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Choose $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.
Thus,

$$\begin{aligned} &\rho(\alpha)(v_1) = -v_1, && \rho(\alpha)(v_2) = v_2, && \rho(\alpha)(v_3) = -v_3, \\ &\rho(\beta)(v_1) = -v_1, && \rho(\beta)(v_2) = -v_2, && \rho(\beta)(v_3) = v_3. \end{aligned}$$

Hence the braiding is diagonal of Cartan type in the basis

$$\begin{aligned} &w_1 = \sigma_1 v_1, && w_2 = \sigma_1 v_2, && w_3 = \sigma_1 v_3, \\ &w_4 = \sigma_2 v_1, && w_5 = \sigma_2 v_2, && w_6 = \sigma_2 v_3, \\ &w_7 = \sigma_3 v_1, && w_8 = \sigma_3 v_2, && w_9 = \sigma_3 v_3; \end{aligned}$$

this implies that the corresponding Dynkin diagram contains the affine Dynkin diagram $A_2^{(1)}$. By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_{2,2,2}, \phi_+) = \infty$. Finally, the case $\rho = \phi_-$ is similar. $\square$

2.3.3. *Case* $n = 4$. Let $\pi = (1\,2)(3\,4)(5\,6)(7\,8)$ in $\mathbb{S}_8$. The isotropy subgroup of π is $\mathbb{S}_8^\pi = < A_1, A_2, A_3, A_4 > \rtimes < B_1, B_2, B_3 > \simeq \mathbb{Z}_2^4 \rtimes \mathbb{S}_4$, where $A_1 = (1\,2)$, $A_2 = (3\,4)$, $A_3 = (5\,6)$, $A_4 = (7\,8)$, and

$$B_1 = (1\,3)(2\,4) \quad B_2 = (3\,5)(4\,6) \quad B_3 = (5\,7)(6\,8).$$

By section 2.3.1, there are 20 irreducible representations of $\mathbb{S}_8^\pi$, but by Remark 2.3 we only need to consider 6 of them. They are

$$\begin{aligned} &\rho_1 = \operatorname{Ind}(\chi_{(1)} \otimes \epsilon), && \rho_2 = \operatorname{Ind}(\chi_{(1)} \otimes \operatorname{sgn}), && \rho_3 = \operatorname{Ind}(\chi_{(1)} \otimes \theta), \\ &\rho_4 = \operatorname{Ind}(\chi_{(3)} \otimes \epsilon), && \rho_5 = \operatorname{Ind}(\chi_{(3)} \otimes \operatorname{sgn}), && \rho_6 = \operatorname{Ind}(\chi_{(3)} \otimes \theta), \end{aligned}$$

where Ind means $\operatorname{Ind}_{\mathbb{Z}_2^4 \rtimes \mathbb{S}_3}^{\mathbb{Z}_2^4 \rtimes \mathbb{S}_4}$ and θ is the standard representation of $\mathbb{S}_3$. With this notation, we can state the following result.

Proposition 2.6. *Let* $\pi \in \mathbb{S}_8$ *of type* (2^4). *Then* $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$ *for any* $\rho \in \widehat{\mathbb{S}_8^\pi}$.

Proof. We can assume that $\pi = (1\,2)(3\,4)(5\,6)(7\,8)$. We shall later prove that if $i = 1, 2, 4$ or 5 then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho_i) = \infty$, for any $n \geq 4$, see Lemmata 2.7 and 2.8. We check now the remaining $i = 3, 6$. It is clear that π, $\alpha = (12)(34)(57)(68)$, $\beta = (12)(34)(58)(67)$ are in $\mathcal{O}_\pi$ and they satisfy the same relations as in the proof of Proposition 2.5, with $\sigma_1 = \mathrm{id}$, $\sigma_2 = (6\,7)$, $\sigma_3 = (6\,8)$. Let us consider ρ_3. In an appropriate basis, we have

$$\rho_3(\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 & -\omega^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\rho_3(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 & -\omega^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega^{-1} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

where $\omega \in \mathbb{C}_3^\times$ is a primitive 3-th root of the unity. It is easy to see that

$$\begin{aligned} v_1 &= e_1 + \omega e_5, & v_2 &= e_2 + \omega e_6, & v_3 &= e_3 + e_4, & v_4 &= e_3 - e_4, \\ v_5 &= e_1 - \omega e_5, & v_6 &= e_2 - \omega e_6, & v_7 &= e_7 + e_8, & v_8 &= e_7 - e_8 \end{aligned}$$

are eigenvectors of eigenvalues 1 or -1. In particular

$$\rho_3(\alpha)v_7 = v_7, \quad \rho_3(\alpha)v_8 = -v_8, \quad \rho_3(\beta)v_7 = -v_7, \quad \rho_3(\beta)v_8 = v_8.$$

So, in the basis $w_1 = \sigma_1 v_7$, $w_2 = \sigma_1 v_8$, $w_3 = \sigma_2 v_7$, $w_4 = \sigma_2 v_8$, $w_5 = \sigma_3 v_7$, $w_6 = \sigma_3 v_8$, the braiding is diagonal of Cartan type. The corresponding Dynkin diagram is not connected; its connected components are $\{1, 4, 6\}$ and $\{2, 3, 5\}$, each of them supporting the affine Dynkin diagram $A_2^{(1)}$. Then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho_3) = \infty$ by Theorem 1.2.

Finally, if $\rho = \rho_6$ we have

$$\rho_6(\alpha) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega^{-1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho_6(\beta) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega^{-1} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The following are eigenvectors of eigenvalues 1 or -1

$$v_1 = e_1 + e_2, \qquad v_2 = e_1 - e_2, \quad v_3 = e_3 + \omega e_7, \quad v_4 = e_4 + \omega e_8,$$
$$v_5 = e_5 + e_6, \qquad v_6 = e_5 - e_6, \quad v_7 = e_3 - \omega e_7, \quad v_8 = e_4 - \omega e_8.$$

In particular

$$\rho_6(\alpha)v_1 = v_1, \quad \rho_6(\alpha)v_2 = -v_2, \quad \rho_6(\beta)v_1 = -v_1, \quad \rho_6(\beta)v_2 = v_2.$$

So, in the basis $w_1 = \sigma_1 v_1$, $w_2 = \sigma_1 v_2$, $w_3 = \sigma_2 v_1$, $w_4 = \sigma_2 v_2$, $w_5 = \sigma_3 v_1$, $w_6 = \sigma_3 v_2$, the braiding is diagonal of Cartan type. This is similar to the case of ρ_3 interchanging the roles of v_7 by v_1 and v_8 by v_2. Then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho_6) = \infty$ by Theorem 1.2. $\square$

2.3.4. *Case n general.* We now begin the analysis of the general case. We prove two lemmata.

Lemma 2.7. *If ρ is either $\operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(1)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n} (\chi_{(1)} \otimes \epsilon)$ or $\operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(1)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n} (\chi_{(1)} \otimes \operatorname{sgn})$ then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, for any $n \geq 2$.*

Proof. Let $\sigma_1 = \mathrm{id}$, $\sigma_2 = (2n-2\, 2n-1)$ and $\sigma_3 = (2n-2\, 2n)$. We define $\alpha := \sigma_2 \triangleright \pi$ and $\beta = \sigma_3 \triangleright \pi$, so

$$\alpha = A_1 A_2 \dots A_{n-2}(2n-3\, 2n-1)(2n-2\, 2n),$$
$$\beta = A_1 A_2 \dots A_{n-2}(2n-3\, 2n)(2n-2\, 2n-1);$$

we set $T = \{\pi, \alpha, \beta\}$. It is straightforward to check that

$$\alpha = \pi A_{n-1} A_n B_{n-1} = \beta A_{n-1} A_n, \qquad \beta = \pi B_{n-1}. \tag{5}$$

Now we proceed as in the proof of Proposition 2.5. Let $\sigma_j v := \sigma_j \otimes v$, $v \in V$, $1 \leq j \leq 3$. The coaction is given by $\delta(\sigma_j v) = \sigma_j \triangleright \pi \otimes \sigma_j v$; the action of the elements π, α, β is

$$\pi \cdot \sigma_1 v = \sigma_1 \rho(\pi)(v), \qquad \pi \cdot \sigma_2 v = \sigma_2 \rho(\alpha)(v), \qquad \pi \cdot \sigma_3 v = \sigma_3 \rho(\beta)(v),$$
$$\alpha \cdot \sigma_1 v = \sigma_1 \rho(\alpha)(v), \qquad \alpha \cdot \sigma_2 v = \sigma_2 \rho(\pi)(v), \qquad \alpha \cdot \sigma_3 v = \sigma_3 \rho(\alpha)(v),$$
$$\beta \cdot \sigma_1 v = \sigma_1 \rho(\beta)(v), \qquad \beta \cdot \sigma_2 v = \sigma_2 \rho(\beta)(v), \qquad \beta \cdot \sigma_3 v = \sigma_3 \rho(\pi)(v).$$

Hence the braiding is given by

$$\begin{aligned}
c(\sigma_1 v \otimes \sigma_2 w) &= \sigma_2\rho(\alpha)(w) \otimes \sigma_1 v, & c(\sigma_1 v \otimes \sigma_3 w) &= \sigma_3\rho(\beta)(w) \otimes \sigma_1 v,\\
c(\sigma_2 v \otimes \sigma_1 w) &= \sigma_1\rho(\alpha)(w) \otimes \sigma_2 v, & c(\sigma_2 v \otimes \sigma_3 w) &= \sigma_3\rho(\alpha)(w) \otimes \sigma_2 v,\\
c(\sigma_3 v \otimes \sigma_1 w) &= \sigma_1\rho(\beta)(w) \otimes \sigma_3 v, & c(\sigma_3 v \otimes \sigma_2 w) &= \sigma_2\rho(\beta)(w) \otimes \sigma_3 v,
\end{aligned}$$

and $c(\sigma_j v \otimes \sigma_j w) = (\sigma_j \triangleright \pi)\cdot \sigma_j w \otimes \sigma_j v = \sigma_j\rho(\pi)(w) \otimes \sigma_j v = -\sigma_j w \otimes \sigma_j v$, for all $1 \le j \le 3$ and $v, w \in V$.

Let us consider $\rho = \operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(1)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n}(\chi_{(1)} \otimes \epsilon)$. The vector space affording ρ has dimension n. It is easy to see that for every i, $1 \le i \le n$, the matrix $\rho(A_i)$ is diagonal with $(\rho(A_i))_{i,i} = -1$ and 1 elsewhere; while

$$\rho(B_{n-1}) = \begin{pmatrix} \mathrm{Id}_{n-2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}.$$

Therefore, we have $\rho(\pi) = -\mathrm{Id}$ and, by (5),

$$\rho(\alpha) = \begin{pmatrix} -\mathrm{Id}_{n-2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} -\mathrm{Id}_{n-2} & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}.$$

Choose $v_i = e_i$, $1 \le i \le n-2$, $v_{n-1} = e_{n-1} + e_n$ and $v_n = e_{n-1} - e_n$. Hence

$$\begin{aligned}
\rho(\alpha)v_i &= -v_i, & \rho(\alpha)v_{n-1} &= v_{n-1}, & \rho(\alpha)v_n &= -v_n,\\
\rho(\beta)v_i &= -v_i, & \rho(\beta)v_{n-1} &= -v_{n-1}, & \rho(\beta)v_n &= v_n,
\end{aligned}$$

with $1 \le i \le n-2$. Then the braiding is diagonal of Cartan type in the basis $\mathcal{B} = \{\sigma_j v_i\}$, $j = 1, 2, 3$, $1 \le i \le n$. The corresponding Dynkin diagram is not of finite type because it contains the affine Dynkin diagram $A_2^{(1)}$. By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$.

Finally, if $\rho = \operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(1)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n}(\chi_{(1)} \otimes \mathrm{sgn})$, $\rho(A_i)$ are as before and

$$\rho(B_{n-1}) = \begin{pmatrix} -\mathrm{Id}_{n-2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}.$$

Then,

$$\rho(\alpha) = \begin{pmatrix} \mathrm{Id}_{n-2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} \mathrm{Id}_{n-2} & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}.$$

Let v_i be as before; hence

$$\begin{aligned} &\rho(\alpha)v_i = v_i, && \rho(\alpha)v_{n-1} = v_{n-1}, && \rho(\alpha)v_n = -v_n,\\ &\rho(\beta)v_i = v_i, && \rho(\beta)v_{n-1} = -v_{n-1}, && \rho(\beta)v_n = v_n, \end{aligned}$$

with $1 \leq i \leq n-2$. Then the braiding is diagonal of Cartan type in the basis $\mathcal{B}$. The corresponding Dynkin diagram is not of finite type because it contains the affine diagram $A_2^{(1)}$. By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$. $\square$

Lemma 2.8. *If ρ is either $\operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(n-1)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n}(\chi_{(n-1)} \otimes \epsilon)$ or $\operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(n-1)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n}(\chi_{(n-1)} \otimes \operatorname{sgn})$ then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$, for any $n \geq 2$.*

Proof. Let σ_j, α and β be as in the proof of Lemma 2.7.

If $\rho = \operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(n-1)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n}(\chi_{(n-1)} \otimes \epsilon)$, then for every i, $1 \leq i \leq n$, the matrix $\rho(A_i)$ is diagonal with $(\rho(A_i))_{n-i+1,n-i+1} = 1$ and -1 elsewhere; while

$$\rho(B_{n-1}) = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \operatorname{Id}_{n-2} \end{pmatrix}.$$

Therefore, we have $\rho(\pi) = -\operatorname{Id}$ and, by (5),

$$\rho(\alpha) = \begin{pmatrix} 0 & -1 & \\ -1 & 0 & \\ & & \operatorname{Id}_{n-2} \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} 0 & -1 & \\ -1 & 0 & \\ & & -\operatorname{Id}_{n-2} \end{pmatrix}.$$

Choose $v_1 = e_1 + e_2$, $v_2 = e_1 - e_2$ and $v_i = e_i$, $3 \leq i \leq n$. Hence

$$\begin{aligned} &\rho(\alpha)v_1 = -v_1, && \rho(\alpha)v_2 = v_2, && \rho(\alpha)v_i = v_i,\\ &\rho(\beta)v_1 = -v_1, && \rho(\beta)v_2 = v_2, && \rho(\beta)v_i = -v_i, \end{aligned}$$

with $3 \leq i \leq n$. Then the braiding is diagonal of Cartan type in the basis $\mathcal{B} = \{\sigma_j v_i\}$, $j = 1,2,3$, $1 \leq i \leq n$. The corresponding Dynkin diagram is not of finite type because it contains the affine Dynkin diagram $A_5^{(1)}$. By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$.

Finally, if $\rho = \operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(n-1)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n}(\chi_{(n-1)} \otimes \operatorname{sgn})$ then the matrices $\rho(A_i)$ are the same as in the previous case and

$$\rho(B_{n-1}) = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -\operatorname{Id}_{n-2} \end{pmatrix}.$$

Then,

$$\rho(\alpha) = \begin{pmatrix} 0 & -1 & \\ -1 & 0 & \\ & & -\operatorname{Id}_{n-2} \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} 0 & -1 & \\ -1 & 0 & \\ & & \operatorname{Id}_{n-2} \end{pmatrix}.$$

Let v_i as before; hence

$$\begin{aligned} &\rho(\alpha)v_1 = -v_1, && \rho(\alpha)v_2 = v_2, && \rho(\alpha)v_i = -v_i, \\ &\rho(\beta)v_1 = -v_1, && \rho(\beta)v_2 = v_2, && \rho(\beta)v_i = v_i, \end{aligned}$$

with $3 \le i \le n$. Then the braiding is diagonal of Cartan type in the basis $\mathcal{B}$. The corresponding Dynkin diagram is not of finite type because it contains the affine Dynkin diagram $A_5^{(1)}$. By Theorem 1.2, $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$. $\square$

Notice that if n is odd, then the analog of Lemma 2.8 follows from Remark 2.3. Next we set some notation separately for the cases n even and n odd.

Notation in case n even. We suppose $n = 2L$ and for every l, with $1 \le l \le L$, we define $\sigma_l^+ := (4l-2\ 4l-1)$, $\sigma_l^- := (4l-2\ 4l)$ and

$$\alpha_l := \sigma_l^+ \triangleright \pi, \qquad \beta_l := \sigma_l^- \triangleright \pi.$$

That is, if $\pi = A_1 A_2 \dots A_{2l-1} A_{2l} \dots A_{2L-1} A_{2L}$, then

$$\begin{aligned} \alpha_l &= A_1 A_2 \dots (4l-3\ 4l-1)(4l-2\ 4l) \dots A_{2L-1} A_{2L}, \\ \beta_l &= A_1 A_2 \dots (4l-3\ 4l)(4l-2\ 4l-1) \dots A_{2L-1} A_{2L}. \end{aligned}$$

It is easy to see that $\sigma_l^\pm \sigma_h^\pm = \sigma_h^\pm \sigma_l^\pm$, for all l, h distinct. Let T be the set

$$T = \{\sigma_{l_k}^\pm \triangleright (\cdots (\sigma_{l_1}^\pm \triangleright \pi) \cdots) \,:\, 1 \le k \le L, 1 \le l_1 < \cdots < l_k \le L\} \cup \{\pi\}.$$

Note that $\sigma_{l_k}^\pm \triangleright (\cdots (\sigma_{l_1}^\pm \triangleright \pi) \cdots) = (\sigma_{l_k}^\pm \cdots \sigma_{l_1}^\pm) \triangleright \pi$. Let $T = \{\pi_0 = \pi, \pi_1, \dots, \pi_N\}$ be a numeration of T; we call $\sigma_0 = \mathrm{id}$ and σ_j the element $\sigma_l^\pm$ such that $\sigma_j \triangleright \pi = \pi_j$, $1 \le j \le N$.

Straightforward computations imply the following.

Lemma 2.9. *For all l, $1 \le l \le L$, we have*

(i) $\sigma_l^- \sigma_l^+ A_{2l-1} A_{2l} \sigma_l^+ \sigma_l^- = \sigma_l^+ A_{2l-1} A_{2l} \sigma_l^+$.
(ii) $\sigma_l^+ \sigma_l^- A_{2l-1} A_{2l} \sigma_l^- \sigma_l^+ = \sigma_l^- A_{2l-1} A_{2l} \sigma_l^-$.
(iii) $\sigma_l^\pm A_{2l-1} A_{2l} \sigma_l^\pm A_{2l-1} A_{2l} = \sigma_l^\mp A_{2l-1} A_{2l} \sigma_l^\mp$.
(iv) $A_{2l-1} A_{2l} \sigma_l^\pm A_{2l-1} A_{2l} \sigma_l^\pm = \sigma_l^\mp A_{2l-1} A_{2l} \sigma_l^\mp$.
(v) $\alpha_l \beta_l = A_{2l-1} A_{2l} = \beta_l \alpha_l$. $\square$

Lemma 2.10. *(1) If $1 \le l \le L$, then*

$$\alpha_l = \pi A_{2l-1} A_{2l} B_{2l-1} = \beta_l A_{2l-1} A_{2l}, \qquad \beta_l = \pi B_{2l-1}.$$

(2) $T \subseteq \mathcal{O}_\pi \cap \mathbb{S}_{2n}^\pi$.
(3) For every i, j, $0 \le i, j \le N$, there exists k, $0 \le k \le N$ such that $\pi_i \sigma_j = \sigma_j \pi_k$.
(4) T is abelian.

Proof. (1) is obvious. (2) Clearly, $T \subseteq \mathcal{O}_\pi$. To see that $T \subseteq \mathbb{S}_{2n}^\pi$ we need to prove that $\pi_i = (\sigma_{l_k}^\pm \cdots \sigma_{l_1}^\pm) \triangleright \pi$ in $\mathbb{S}_{2n}^\pi$. This is clear for $k = 1$; then it follows by induction on k.

(3) Note that if $i = 0$ then $k = j$; if $i = j$ then $k = 0$., etc. Fix $\pi_i = (\sigma_{l_k}^\pm \cdots \sigma_{l_1}^\pm) \triangleright \pi$, with $1 \le l_1 < \cdots < l_k \le L$; suppose $\sigma_j = \sigma_{h_M}^\pm \cdots \sigma_{h_1}^\pm$, with $1 \le h_1 < \cdots < h_M \le L$. Then

$$\sigma_j \pi_i \sigma_j = \sigma_{l_k}^\pm \cdots \sigma_{l_1}^\pm \sigma_{h_M}^\pm \cdots \sigma_{h_1}^\pm \pi \sigma_{h_1}^\pm \cdots \sigma_{h_M}^\pm \sigma_{l_1}^\pm \cdots \sigma_{l_k}^\pm.$$

If $l_r \ne h_s$ for all r, s then $\pi_k := \sigma_j \pi_i \sigma_j$ is in T. If $l_r = h_s$ for many r, s we have that in the expression $\sigma_j \pi_i \sigma_j$ the factors $\sigma_{l_r}^\pm$ and $\sigma_{h_s}^\pm$ cancel mutually while for the factors $\sigma_{l_r}^\pm$ and $\sigma_{h_s}^\mp$ we use the Lemma 2.9(i),(ii) and we have

$$\sigma_j \pi_i \sigma_j = \cdots \sigma_{l_r}^\pm \sigma_{h_s}^\mp \pi \sigma_{h_s}^\mp \sigma_{l_r}^\pm \cdots = \cdots \sigma_{h_s}^\mp \pi \sigma_{h_s}^\mp \cdots.$$

Therefore $\sigma_j \pi_i \sigma_j$ is in T.

(4) We need to prove $\pi_i \pi_j = \pi_j \pi_i$, for every i, j.

(a) We analyze the cases when $\pi_i = \alpha_l$ or β_l and $\pi_j = \alpha_h$ or β_h.

Case (i). If $\pi_i = \alpha_l$ and $\pi_j = \alpha_h$, then

$$\pi_i \pi_j = \sigma_l^+ A_{2l-1} A_{2l} \sigma_l^+ \sigma_h^+ A_{2h-1} A_{2h} \sigma_h^+.$$

If $l = h$ then the claim is clearly true. If $l \ne h$ we have that $\sigma_l^+ A_{2l-1} A_{2l} \sigma_l^+$ and $\sigma_h^+ A_{2h-1} A_{2h} \sigma_h^+$ commute because they are disjoint permutations; hence the result follows.

Case (ii). If $\pi_i = \beta_l$ and $\pi_j = \beta_h$. Idem.

Case (iii). If $\pi_i = \alpha_l$ and $\pi_j = \beta_h$, then

$$\pi_i \pi_j = \sigma_l^+ A_{2l-1} A_{2l} \sigma_l^+ \sigma_h^- A_{2h-1} A_{2h} \sigma_h^-.$$

If $l \ne h$ then $\sigma_l^+ A_{2l-1} A_{2l} \sigma_l^+$ and $\sigma_h^- A_{2h-1} A_{2h} \sigma_h^-$ commute; hence $\pi_i \pi_j = \pi_j \pi_i$. While if $l = h$, it is easy to check that $\pi_i \pi_j = \alpha_l \beta_l = \mathrm{id} = \beta_l \alpha_l = \pi_j \pi_i$, and the result follows.

(b) In general, for $\pi_i = (\sigma_{l_k}^{\pm} \cdots \sigma_{l_1}^{\pm}) \triangleright \pi$ and $\pi_j = (\sigma_{h_M}^{\pm} \cdots \sigma_{h_1}^{\pm}) \triangleright \pi$, we have

$$\pi_i = A_1 A_2 \cdots \sigma_{l_1}^{\pm} A_{2l_1-1} A_{2l_1} \sigma_{l_1}^{\pm} \cdots \sigma_{l_k}^{\pm} A_{2l_k-1} A_{2l_k} \sigma_{l_k}^{\pm} \cdots A_{2L-1} A_{2L},$$
$$\pi_j = A_1 A_2 \cdots \sigma_{h_1}^{\pm} A_{2h_1-1} A_{2h_1} \sigma_{h_1}^{\pm} \cdots \sigma_{h_M}^{\pm} A_{2h_M-1} A_{2h_M} \sigma_{h_M}^{\pm} \cdots A_{2L-1} A_{2L}.$$

We have two cases:
(i) If $l_r \neq h_s$, for all r, s. Then

$$\begin{aligned}\pi_i \pi_j = \sigma_{l_1}^{\pm} A_{2l_1-1} A_{2l_1} \sigma_{l_1}^{\pm} \dots \sigma_{l_k}^{\pm} A_{2l_k-1} A_{2l_k} \sigma_{l_k}^{\pm} \\ \sigma_{h_1}^{\pm} A_{2h_1-1} A_{2h_1} \sigma_{h_1}^{\pm} \dots \sigma_{h_M}^{\pm} A_{2h_M-1} A_{2h_M} \sigma_{h_M}^{\pm} = \pi_j \pi_i,\end{aligned}$$

because every $\sigma_{l_r}^{\pm} A_{2l_r-1} A_{2l_r} \sigma_{l_r}^{\pm}$ commute with every $\sigma_{h_s}^{\pm} A_{2h_s-1} A_{2h_s} \sigma_{h_s}^{\pm}$.
(ii) If $l_r = h_s$, for some r, s, we use (a) in every factor corresponding to $l_r = h_s$. $\square$

For the rest of this subsection we fix the order in T given by

$$T = \{\pi_0 = \pi, \pi_1 = \alpha_1, \pi_2 = \beta_1, \dots, \pi_{2L-1} = \alpha_L, \pi_{2L} = \beta_L, \dots\}.$$

Next we deal with $\rho = \rho_{\chi(k),\mu}$ in $\widehat{\mathbb{S}_{2n}^{\pi}}$, as in Section 2.3.1; as usual, let V be the vector space affording ρ and V_μ the vector space affording μ. By Remark 2.3, we only need to consider k odd; thus $\rho(\pi) = -\operatorname{Id}$. Since $\pi_i^2 = \mathrm{id}$, for all i, then the possibles eigenvalues of the operators $\{\rho(\pi_i) : 0 \le i \le N\}$ are 1 and -1. Moreover, since T is abelian there exists a basis $\mathcal{B}$ of V of simultaneous eigenvectors– say $\mathcal{B} = \{v_1, \dots, v_R\}$. Note that

$$\dim V = [\mathbb{S}_n : \mathbb{S}_n^{\chi(k)}] \dim V_\mu = \binom{n}{k} \dim V_\mu.$$

For every i, $0 \le i \le N$ we define $\mathbf{f}^i = (f_1^i, f_2^i, \dots, f_R^i)$ where $\rho(\pi_i) v_r = f_r^i v_r$, $1 \le r \le R$; for instance $\mathbf{f}^0 = (-1, -1, \dots, -1)$. Now we denote by E_i the matrix with all its rows equal to $\mathbf{f}^i$. Hence E_0 is the matrix $\dim V \times \dim V$ with all its entries equal to -1.

Let us consider the subspace W of $M(\mathcal{O}_\pi, \rho)$ with basis $\{w_{i,r} := \sigma_i v_r = \sigma_i \otimes v_r : 0 \le i \le N, 1 \le r \le R\}$. Then W is a braided vector subspace of Cartan type and the matrix of the scalars $(q_{a,b})_{a,b}$– see section 1.2– has the form

$$\mathcal{Q} = \begin{pmatrix} E_0 & E_1 & E_2 & \cdots & E_{2L-1} & E_{2L} & \cdots \\ E_1 & E_0 & E_1 & \cdots & \cdots & \cdots & \cdots \\ E_2 & E_2 & E_0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ E_{2L-1} & \cdots & \cdots & \cdots & E_0 & E_{2L-1} & \cdots \\ E_{2L} & \cdots & \cdots & \cdots & E_{2L} & E_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here the diagonal blocks are equal to the matrix E_0; whereas the block in the position i, j is the matrix E_k where $\pi_i \sigma_j = \sigma_j \pi_k$.

Notation in case n odd. We suppose $n = 2L+1$ and for every l, with $1 \leq l \leq L$, we take σ_l^+, σ_l^-, α_l, β_l and T as for n even. So

$$\begin{aligned}\pi &= A_1 A_2 \dots A_{2l-1} A_{2l} \dots A_{2L-1} A_{2L} A_{2L+1},\\ \alpha_l &= A_1 A_2 \dots (4l-3\ 4l-1)(4l-2\ 4l) \dots A_{2L-1} A_{2L} A_{2L+1},\\ \beta_l &= A_1 A_2 \dots (4l-3\ 4l)(4l-2\ 4l-1) \dots A_{2L-1} A_{2L} A_{2L+1}.\end{aligned}$$

Then $\sigma_l^{\pm}$,α_l, β_l, π_i, σ_j and T satisfy the same properties as before.

Proposition 2.11. *Let $\rho = (\rho, V)$ be in $\widehat{\mathbb{S}_{2n}^{\pi}}$, $n \geq 2$. If*

(a) *n is even, or*
(b) *$n = 3$ and $\rho \neq \chi_{-,\pm}$, or*
(c) *n is odd and $\rho \neq \rho_{\chi_{(n)},\mu}$, for any μ in $\widehat{\mathbb{S}_n}$,*

then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$.

Proof. Let $\rho = \rho_{\chi_{(k)},\mu}$, $\chi_{(k)}$ in $\widehat{\mathbb{Z}_2^n}$ and μ in $\widehat{\mathbb{S}_n}$. By Remark 2.3, we can assume that k is odd. We distinguish two possibilities.

(1) For every l, with $1 \leq l \leq L$, $\rho(\alpha_l) = \mathrm{Id} = \rho(\beta_l)$ or $\rho(\alpha_l) = -\mathrm{Id} = \rho(\beta_l)$. By Lemma 2.10 (1), this implies

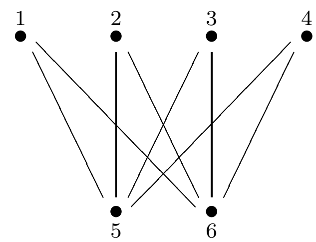
$$\rho(A_{2l-1}A_{2l}) = \mathrm{Id}, \qquad \text{for all } l.$$

Assume that n is even. Hence $\rho(\pi) = \rho(A_1 A_2 \cdots A_{2L-1} A_{2L}) = \mathrm{Id}$; so $q_{\pi,\pi} = 1$ and $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$.

Assume that n is odd. Since $\rho(\pi) = -\mathrm{Id}$, it is easy to see that $\rho(A_{2L+1}) = -\mathrm{Id}$. By the discussion in subsection 2.3.1, this implies $\rho(A_j) = -\mathrm{Id}$, $1 \leq j \leq 2L+1$. Then $\rho = \rho_{\chi_{(n)},\mu}$, but this is a contradiction by hypothesis.

(2) There exists l with $\rho(\alpha_l) \neq \pm\mathrm{Id}$ or $\rho(\beta_l) \neq \pm\mathrm{Id}$ or $\rho(\alpha_l) = \pm\mathrm{Id} = \mp\rho(\beta_l)$. Here we have that if $\dim V > 4$ then the generalized Cartan matrix $\mathcal{A}$ is such that its associated Dynkin diagram is not of finite type and the result follows. For see this we suppose that there exists l with $\rho(\alpha_l) \neq \pm\mathrm{Id}$; for the other cases the argument is similar. We regard that the components of the vector $\mathbf{f}^l$ are 1 or -1; so we define $c^+ := \mathrm{card}\{r : f_r^l = 1\}$ and $c^- := \mathrm{card}\{r : f_r^l = -1\}$; note that $c^+ + c^- = R$. We consider three cases.
(i) If $\dim V \geq 7$ then the associated Dynkin diagram has a vertex w with $\lambda_w \geq 4$, where λ_w denotes the number of vertices of the diagram which are adjacent to w. Hence, such diagram is not of finite type.
(ii) Let $\dim V = 6$; if $c^+ \geq 4$ or $c^- \geq 4$ we proceed as in (i). So, we must consider $c^+ \leq 3$ and $c^- \leq 3$. Because $c^+ + c^- = 6$ then $c^+ = 3$ and $c^- = 3$, but since there is no Dynkin diagram of finite type with two vertices w, w' with $\lambda_w = 3$ and $\lambda_{w'} = 3$, the result follows.

(iii) If $\dim V = 5$ we only must consider either $c^+ = 3$ and $c^- = 2$ or $c^+ = 2$ and $c^- = 3$, by (ii). In any case we have that the associated Dynkin diagram has two vertices w, w' with $\lambda_w = 3$ and $\lambda_{w'} = 3$ and the result follows.



Thus, we must consider (ρ, V) with $\dim V \leq 4$. Then, since $\dim V = \binom{n}{k} \dim V_\mu$, where V_μ is the vector space affording μ, we must consider the different possibilities for n, k and μ which satisfy the condition

$$\binom{n}{k} \dim V_\mu \leq 4.$$

This inequality holds only in the following cases

(i) $n = 2$ and $k = 1$.
(ii) $n = 3$, $k = 1$ or 2 and hence $\dim V_\mu = 1$.
(iii) $n = 4$, $k = 1$ or 3 and hence $\dim V_\mu = 1$.
(iv) any n, $k = 0$ or $k = n$ and $\dim V_\mu = 1, 2, 3$ or 4.

In (i), (ii) and (iii) the result follows from [AZ, Th. 2.7], Propositions 2.5 and 2.6, respectively. In the case (iv), $k \neq 0$ by Remark 2.3 and $k = n$ would be considered for n odd, but this was discarded by hypothesis. $\square$

Theorem 2.12. *Let $\pi \in \mathbb{S}_{2n}$ of type (2^n).*

(a). If n is even then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$ for any $\rho \in \widehat{\mathbb{S}_{2n}^{\pi}}$.

(b). If n is odd and $\rho \neq \chi_{(n)} \otimes \epsilon$, $\chi_{(n)} \otimes \operatorname{sgn}$, then $\dim \mathfrak{B}(\mathcal{O}_\pi, \rho) = \infty$ for any $\rho \in \widehat{\mathbb{S}_{2n}^{\pi}}$.

The braided vector spaces associated to $\chi_{(n)} \otimes \epsilon$ or to $\chi_{(n)} \otimes \operatorname{sgn}$ were considered in Proposition 2.4.

Proof. We can assume that $\pi = (1\,2)(3\,4)\dots(2n-1\,2n)$. By Propositions 2.5 and 2.11, we only need to consider $3 < n$ odd and $\rho = \rho_{\chi_{(n)}, \mu}$, with μ in $\widehat{\mathbb{S}_n}$, $\mu \neq \varepsilon, \operatorname{sgn}$. Notice that

$$\rho = \operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n^{\chi_{(n)}}}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n} (\chi_{(n)} \otimes \mu) = \operatorname{Ind}_{\mathbb{Z}_2^n \rtimes \mathbb{S}_n}^{\mathbb{Z}_2^n \rtimes \mathbb{S}_n} (\chi_{(n)} \otimes \mu) = \chi_{(n)} \otimes \mu.$$

We distinguish two possibilities, as in the proof of 2.11.

(1). We suppose $\rho(\alpha_l) = \operatorname{Id} = \rho(\beta_l)$ or $\rho(\alpha_l) = -\operatorname{Id} = \rho(\beta_l)$, for every l, with $1 \leq l \leq L$. Then it is easy to check that $\rho(B_{2l-1}) = \pm \operatorname{Id}$, $1 \leq l \leq L$.

Since $B_2, B_4, \dots, B_{2L}$ are disjoint permutations we have that the operators $\rho(B_j)$, $1 \leq j \leq n$, commute. Hence, there exists a basis of simultaneous eigenvectors of such operators. This says that the representation μ is not irreducible unless $\dim V_\mu = 1$, and therefore $\mu = \operatorname{sgn}$, but this is a contradiction by hypothesis. The case $\rho(\alpha_l) = -\operatorname{Id} = \rho(\beta_l)$, $1 \leq l \leq L$, implies $\rho(B_{2l-1}) = \operatorname{Id}$, $1 \leq l \leq L$; by analogous arguments we conclude $\mu = \epsilon$, a contradiction by hypothesis.

(2). If $n \geq 7$ and $\mu \neq \epsilon, \operatorname{sgn}$, then $\dim V_\mu > 4$, see [FH, 4.14]; hence $\dim \mathfrak{B}(\mathcal{O}_\pi^{2n}, \rho) = \infty$. It remains the case $n = 5$ and the representations

$$\rho = \chi_{(5)} \otimes \phi, \qquad \rho = \chi_{(5)} \otimes \psi,$$

where ϕ, ψ are the two irreducible representations of $\mathbb{S}_5$ of dimension 4, let us say ϕ the standard representation of $\mathbb{S}_5$ and ψ its conjugated representation.

Let us consider $\rho = \chi_{(5)} \otimes \phi$. We take $\pi = A_1A_2A_3A_4A_5$, B_j, $\sigma^{\pm}$, α_l, β_l, π_i, σ_j and T as in the case n odd; so $\mathbb{S}_{10}^\pi =< A_1, A_2, A_3, A_4, A_5 > \rtimes < B_1, B_2, B_3, B_4 > \simeq \mathbb{Z}_2^5 \rtimes \mathbb{S}_5$ and $T = \{\pi_0 = \pi, \pi_1, \dots, \pi_8\}$ satisfying

$$\pi_1 = B_1A_3A_4A_5,\ \pi_2 = \pi B_1,\ \pi_3 = A_1A_2B_3A_5,\ \pi_4 = \pi B_3,\ \pi_5 = B_1B_3A_5,$$

$$\pi_6 = B_1A_3A_4B_3A_5,\ \pi_7 = A_1A_2B_1B_3A_5,\ \pi_8 = A_1A_2B_1A_3A_4B_3A_5.$$

It is easy to check that the standard representation of $\mathbb{S}_5$ can be given by

$$\phi(1\,2) = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \phi(2\,3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\phi(3\,4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \phi(4\,5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then it is clear that

$$\rho(\pi) = -\operatorname{Id}, \quad \rho(\pi_1) = \rho(\pi_2) = -\phi(B_1), \quad \rho(\pi_3) = \rho(\pi_4) = -\phi(B_3),$$

$$\rho(\pi_5) = \rho(\pi_6) = \rho(\pi_7) = \rho(\pi_8) = -\phi(B_1)\phi(B_3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

If $v_1 = (1, 0, 0, 0)$, $v_2 = (0, 1, -1, 0)$, $v_3 = (1, 0, 0, -2)$ and $v_4 = (1, 1, 1, -4)$ then they are simultaneous eigenvectors of those operators. Hence, we have

$\mathbf{f}^1 = \mathbf{f}^2 = (1,-1,-1,-1)$, $\mathbf{f}^3 = \mathbf{f}^4 = (-1,1,-1,-1)$ and $\mathbf{f}^5 = \mathbf{f}^6 = \mathbf{f}^7 = \mathbf{f}^8 = (1,1,-1,-1)$. Thus, in the basis

$$\begin{array}{lll} w_1 = \sigma_0 v_1, & w_2 = \sigma_0 v_2, & w_3 = \sigma_1 v_1 \\ w_4 = \sigma_1 v_2, & w_5 = \sigma_2 v_1, & w_6 = \sigma_2 v_2, \end{array}$$

the braiding is diagonal of Cartan type and the matrix $\mathcal{Q}$ of the scalars $(q_{a,b})_{a,b}$ is

$$\mathcal{Q} = \begin{pmatrix} -1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix};$$

the corresponding Dynkin diagram is $A_5^{(1)}$, which is not of finite type. Hence $\dim \mathfrak{B}(\mathcal{O}_\pi^{10}, \rho) = \infty$.

Finally, if $\rho = \chi_{(5)} \otimes \psi$ we proceed as the previous case using that the representation ψ is given by $\psi = \operatorname{sgn} \times \phi$. So, in the same basis as before we have that the braiding is diagonal of Cartan type and we obtain the same matrix $\mathcal{Q}$; hence the result follows. $\square$

2.4. **Nichols algebras corresponding to even unmixed permutations.** Let $r, n \in \mathbb{N}$, $r, n \geq 2$. Let $\pi = A_1 \dots A_n$ in $\mathbb{S}_{2rn}$, where A_j is the $2r$-cycle

$$A_j = (2rj - 2r + 1 \quad 2rj - 2r + 2 \quad \cdots \quad 2rj),$$

for every j, $1 \leq j \leq n$. As explained in section 2.1, we have

$$\mathbb{S}_{2rn}^\pi = \langle A_1, \dots, A_n \rangle \rtimes \langle B_1, \dots, B_{n-1} \rangle \simeq \mathbb{Z}_{2r}^n \rtimes \mathbb{S}_n, \tag{6}$$

where B_i is the involution

$$B_i = (2r(i-1)+1 \;\; 2ri+1)\,(2r(i-1)+2 \;\; 2ri+2) \cdots (2ri \;\; 2r(i+1)),$$

$1 \leq i \leq n-1$. Then A_j and B_i satisfy the relations analogous to those in subsection 2.3. Let ρ be an irreducible representation of $\mathbb{S}_{2rn}^\pi$ of the form

$$\rho = \operatorname{Ind}_{\mathbb{Z}_{2r}^n \rtimes \mathbb{S}_n^\chi}^{\mathbb{Z}_{2r}^n \rtimes \mathbb{S}_n} (\chi \otimes \mu), \tag{7}$$

where $\chi \in \widehat{\mathbb{Z}_{2r}^n}$ and $\mu \in \widehat{\mathbb{S}_n^\chi}$. Let $\omega = \exp(\frac{i\pi}{r}) \in \mathbb{G}_{2r}$ a primitive $2r$-th root of 1; any irreducible representation of $\mathbb{Z}_{2r}^n$ is isomorphic to $\chi_{u_1,\dots,u_n}$, where

$$\chi_{u_1,\dots,u_n}(A_j) = \omega^{u_j}, \quad 1 \leq j \leq n, \tag{8}$$

with $0 \leq u_j \leq 2r-1$.

Notation: if ρ is as in (7), with χ as in (8), we shall write $\rho = \rho_{u_1,\dots,u_n,\mu}$.

By Lemma 2.1, if $\rho(\pi) \neq -\operatorname{Id}$, then $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$. Hence, in the following we only consider $\rho = \rho_{\chi_{u_1,\dots,u_n},\mu}$ such that $\rho(\pi) = -\operatorname{Id}$; that is

$$\omega^{u_1+\cdots+u_n} = -1, \tag{9}$$

i.e. $u_1 + \cdots + u_n = r, 3r, 5r, \dots, (2n-1)r$.

For every (i, j), with $1 \leq i < j \leq n$, we define

$$B_{ij} := \begin{cases} B_i & , \text{ if } |i-j| = 1, \\ B_i B_{i+1} \cdots B_{j-1} \cdots B_{i+1} B_i & , \text{ if } |i-j| > 1, \end{cases}$$

and $\pi_{(i,j)} := \pi B_{ij}$. Note that B_{ij} acts as the transposition $(i\ \ j)$ on $A_1, \dots, A_n$ and that $\pi_{(i,j)}$ is in $\mathbb{S}^{\pi}_{2rn}$. We can state the following.

Lemma 2.13. *For every (i, j), with $1 \leq i < j \leq n$, we have*

(a) *$\pi_{(i,j)}$ is in $\mathcal{C}$.*
(b) *there exists an involution $\sigma_{(i,j)}$ such that $\pi_{(i,j)} = \sigma_{(i,j)} \pi \sigma_{(i,j)}$.*
(c) *there exist involutions σ, $\widetilde{\sigma}_{(i,j)}$ in $\mathbb{S}_{2rn}$ such that $\pi^{-1} = \sigma\pi\sigma$ and $\pi^{-1}_{(i,j)} = \widetilde{\sigma}_{(i,j)} \pi \widetilde{\sigma}_{(i,j)}$.*

Proof. It is enough to prove this for $i = 1$ and $j = 2$.

(a) and (b). It is easy to see that

$$\begin{aligned} \pi_{(1,2)} = \pi B_1 = (1\ \ 2r+2\ \ 3\ \ 2r+4\ \ 5 \dots 4r-2\ \ 2r-1\ \ 4r) \\ \times (2\ \ 2r+3\ \ 4\ \ 2r+5\ \ 6 \dots 4r-1\ \ 2r\ \ 2r+1) A_3 \cdots A_n, \end{aligned}$$

and we can choose

$$\sigma_{(1,2)} = (2\ \ 2r+2)(4\ \ 2r+4) \cdots (2r-2\ \ 4r-2)(2r\ \ 4r).$$

Clearly, $\sigma_{(1,2)}$ is an involution and $\pi_{(1,2)} = \sigma_{(1,2)} \pi \sigma_{(1,2)} \in \mathcal{C}$.

(c) For every j, $1 \leq j \leq n$,

$$\sigma_j := \prod_{h=1}^{r} (2(j-1)r + h\ \ 2jr - h + 1)$$

is an involution and satisfies $\sigma_j \pi \sigma_j = A_1 \cdots A_j^{-1} \cdots A_n$. Now, if $\sigma = \sigma_1 \cdots \sigma_n$ then $\pi^{-1} = \sigma\pi\sigma$. Finally, *if r is even* and

$$\begin{aligned} \widetilde{\sigma}_{(1,2)} = &(2\ \ 4r)(4\ \ 4r-2) \cdots (2r\ \ 2r+2) \\ &(3\ \ 2r-1)(5\ \ 2r-3) \cdots (r-3\ \ r+5)(r-1\ \ r+3) \\ &(2r+3\ \ 4r-1)(2r+5\ \ 4r-3) \cdots (2r+r-1\ \ 2r+r+3), \end{aligned}$$

or *if* r *is odd* and

$$\begin{aligned}\widetilde{\sigma}_{(1,2)} =&(2\ \ 4r)(4\ \ 4r-2)\cdots(2r\ \ 2r+2)\\ &(3\ \ 2r-1)(5\ \ 2r-3)\cdots(r-2\ \ r+4)(r\ \ r+2)\\ &(2r+3\ \ 4r-1)(2r+5\ \ 4r-3)\cdots(2r+r\ \ 2r+r+2),\end{aligned}$$

then $\widetilde{\sigma}^2_{(1,2)} = \mathrm{id}$, and straightforward computations imply that $\pi^{-1}_{(1,2)} = \widetilde{\sigma}_{(1,2)}\pi\widetilde{\sigma}_{(1,2)}$. □

We now consider two different cases according to the degree of ρ.

2.4.1. *The degree of ρ is greater that 1.*

Theorem 2.14. *Let ρ be in $\widehat{\mathbb{S}^{\pi}_{2rn}}$. If $\deg\rho > 1$ then $\dim\mathfrak{B}(\mathcal{C},\rho) = \infty$.*

Proof. Let us consider two possibilities.

(A) *Assume that there exists (i,j), with $1 \le i < j \le n$, such that* $\rho\left(\pi_{(i,j)}\right) \ne \pm\,\mathrm{Id}$. For simplicity, we denote

$$\begin{aligned}&t_1 := \pi, && t_2 := \pi^{-1}, && t_3 := \pi_{(i,j)}, && t_4 := \pi^{-1}_{(i,j)},\\ &g_1 := \mathrm{id}, && g_2 := \sigma, && g_3 := \sigma_{(i,j)}, && g_4 := \widetilde{\sigma}_{(i,j)}.\end{aligned}$$

Now, we have the following relations: $t_1 g_l = g_l t_l$, $l = 1,2,3,4$, and

$$\begin{aligned}&t_2g_1 = g_1t_2, && t_2g_2 = g_2t_1, && t_2g_3 = g_3t_4, && t_2g_4 = g_4t_3,\\ &t_3g_1 = g_1t_3, && t_3g_2 = g_2t_4, && t_3g_3 = g_3t_1, && t_3g_4 = g_4t_2,\\ &t_4g_1 = g_1t_4, && t_4g_2 = g_2t_3, && t_4g_3 = g_3t_2, && t_4g_4 = g_4t_1.\end{aligned}$$

Since the elements t_1, t_2, t_3 and t_4 commute then there exists a basis of simultaneous eigenvectors $\{v_1,\dots,v_R\}$ of V, the vector space affording ρ. Hence, either the operator $\rho\left(\pi_{(i,j)}\right)$ has at least two distinct eigenvalues or $\rho\left(\pi_{(i,j)}\right) = \lambda\,\mathrm{Id}$, with $\lambda \ne \pm 1$.

In the first case, there exist s and s', $1 \le s, s' \le R$, such that

$$\rho(\pi_{(i,j)})\, v_s = \lambda_s v_s \quad \text{and} \quad \rho(\pi_{(i,j)})\, v_{s'} = \lambda_{s'} v_{s'},$$

with $\lambda_s \ne \lambda_{s'}$; let us consider the subspace W of $M(\mathcal{C},\rho)$ generated by

$$\{g_1v_s,\, g_1v_{s'},\, g_2v_s,\, g_2v_{s'},\, g_3v_s,\, g_3v_{s'},\, g_4v_s,\, g_4v_{s'}\}. \tag{10}$$

It is clear that W is a braided vector subspace of diagonal type of $M(\mathcal{C},\rho)$. Now, if $\lambda_s^2 \ne 1$ then it is easy to see that the generalized Dynkin diagram

contains a cycle of the form

$$\begin{tikzcd} & \overset{-1}{\bullet} \arrow[dl, dash, "\lambda_s^2"'] \arrow[dr, dash, "\lambda_s^{-2}"] & \\ -1\bullet & & \bullet -1 \\ & \underset{-1}{\bullet} \arrow[ul, dash, "\lambda_s^{-2}"] \arrow[ur, dash, "\lambda_s^2"'] & \end{tikzcd}\ ; \tag{11}$$

while that if $\lambda_s^2 = 1$ then $\lambda_s \lambda_{s'} \neq 1$, this implies that the generalized Dynkin diagram contains a cycle of the form

$$\begin{tikzcd} & \overset{-1}{\bullet} \arrow[dl, dash, "\lambda_s\lambda_{s'}"'] \arrow[dr, dash, "\lambda_s^{-1}\lambda_{s'}^{-1}"] & \\ -1\bullet & & \bullet -1 \\ & \underset{-1}{\bullet} \arrow[ul, dash, "\lambda_s^{-1}\lambda_{s'}^{-1}"] \arrow[ur, dash, "\lambda_s\lambda_{s'}"'] & \end{tikzcd}\ .$$

Hence, in both cases we have $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$, by [H4].

In the second case, we choose any s, $1 \leq s \leq R$; then the subspace of $M(\mathcal{C}, \rho)$ generated by

$$\{g_1 v_s,\ g_2 v_s,\ g_3 v_s,\ g_4 v_s\},$$

is a braided vector subspace of diagonal type of $M(\mathcal{C}, \rho)$, and its Dynkin diagram contains a cycle as in (11). Hence, $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$, by [H4].

(B) *Assume that* $\rho\left(\pi_{(i,j)}\right) = \pm \operatorname{Id}$, *for every* (i, j), *with* $1 \leq i < j \leq n$. The relation $\pi_{(1,2)} = \pi B_1$ gives $\rho(B_1) = \mp \operatorname{Id}$; the relations $\pi_{(1,3)} = \pi B_1 B_2 B_1$ and $\rho(B_1) = \pm \operatorname{Id}$ imply that $\rho(B_2) = \mp \operatorname{Id}$, and so on. Hence, the operators $\rho(A_1)$, $\dots$, $\rho(A_n)$, $\rho(B_1)$, $\dots$, $\rho(B_{n-1})$ commute, and there exists a basis of simultaneous eigenvectors of V for those operators. Since $\deg \rho > 1$, ρ is not an irreducible representation of $\mathbb{S}^{\pi}_{2rn}$, which is a contradiction. $\square$

2.4.2. *The degree of* ρ *is 1.* Say $V = \mathbb{C}$ - span of v. By (7), $\deg \rho = [\mathbb{S}_n : \mathbb{S}_n^{\chi}]\ \deg \mu$; thus $\mathbb{S}_n^{\chi} = \mathbb{S}_n$ and $\deg \mu = 1$. This implies that $\rho = \chi_{c,\dots,c} \otimes \mu$, for some c, with $0 \leq c \leq 2r - 1$, and $\mu = \epsilon$ or sgn. Note that if $c = 0$ then $\rho(\pi) = 1$, which is a contradiction by hypothesis. So, we can assume $c \neq 0$.

We begin by the following result.

Proposition 2.15. *Let $\rho = \chi_{c,\dots,c} \otimes \mu$, with $0 < c \le 2r-1$.*

(a) *If r is odd and $c \neq r$, then $\dim \mathfrak{B}(\mathcal{C},\rho) = \infty$.*
(b) *If r is even and $c \neq \frac{r}{2}$, r, $\frac{3r}{2}$, then $\dim \mathfrak{B}(\mathcal{C},\rho) = \infty$.*

Proof. Let

$$t_1 := \pi, \qquad t_2 := \pi^{-1}, \qquad t_3 := A_1^{-1}A_2\cdots A_n, \qquad t_4 := t_3^{-1},$$
$$g_1 := \mathrm{id}, \qquad g_2 := \sigma, \qquad g_3 := \sigma_1, \qquad g_4 := \sigma_2\cdots\sigma_n,$$

where $\sigma_1, \dots, \sigma_n$ are as in the proof of Lemma 2.13 (c). It is clear that they satisfy the same relations as in subsection 2.4.1. Then the subspace of $M(\mathcal{C},\rho)$ generated by $\{g_1v, g_2v, g_3v, g_4v\}$ is braided of diagonal type which matrix of coefficients $(q_{ij})_{ij}$, see subsection 1.2, given by

$$\mathcal{Q} = \begin{pmatrix} -1 & -1 & -\omega^{-2c} & -\omega^{2c} \\ -1 & -1 & -\omega^{2c} & -\omega^{-2c} \\ -\omega^{-2c} & -\omega^{2c} & -1 & -1 \\ -\omega^{2c} & -\omega^{-2c} & -1 & -1 \end{pmatrix}.$$

Since $c \le 2r-1$, it is easy to see that $\omega^{4c} = 1$ if and only if $2c = r$, $2r$ or $3r$. Now, it is clear that if r is odd and $c \neq r$, or if r is even and $c \neq \frac{r}{2}$, r, $\frac{3r}{2}$, we have that $\omega^{4c} \neq 1$. This implies that the generalized Dynkin diagram has a cycle as in (11). Hence, $\dim \mathfrak{B}(\mathcal{C},\rho) = \infty$. $\square$

In the remaining cases, the braiding is always negative.

Theorem 2.16. *Assume that $\rho(\pi) = -1$.*
(a) If r is odd and $\rho = \chi_{r,\dots,r} \otimes \mu$, with $\mu = \epsilon$ or sgn, then the braiding is negative.
(b) If r is even and $\rho = \chi_{c,\dots,c} \otimes \mu$, with $c = \frac{r}{2}$, r or $\frac{3r}{2}$ and $\mu = \epsilon$ or sgn, then the braiding is negative.

Note that, for ρ as in (a) or (b), $\rho(\pi)$ is not necessarily equal to -1.

In order to prove this result, we need two lemmata. Let us remember that $t_1 = \pi, \dots, t_M$ is a numeration of $\mathcal{C}$ and $g_l \in \mathbb{S}_{2rn}$ are such that $g_l\pi g_l^{-1} = t_l$, for all $1 \le l \le M$; we choose $g_1 = \mathrm{id}$.

Let t_l in $\mathcal{C}$, such that $\pi t_l = t_l\pi$, i.e. t_l in $\mathbb{S}_{2rn}^{\pi}$. We know that $\gamma_{l1} := g_1^{-1}t_lg_1 = t_l$ and $\gamma_{1l} := g_l^{-1}\pi g_l$ are in $\mathcal{C} \cap \mathbb{S}_{2rn}^{\pi}$. By (6), we can write

$$\gamma_{l1} = A_1^{d_1}\cdots A_n^{d_n}B, \tag{12}$$
$$\gamma_{1l} = A_1^{e_1}\cdots A_n^{e_n}B', \tag{13}$$

where B and B' are in $\langle B_1,\dots,B_{n-1}\rangle \simeq \mathbb{S}_n$. Let $\Phi : \langle B_1,\dots,B_{n-1}\rangle \to \mathbb{S}_n$ be the group isomorphism given by $\Phi(B_i) = (i\ i+1)$, $1 \le i \le n-1$.

For every j, $1 \le j \le n$, we define

$$\mathbb{A}_j := \{2rj - 2r + 1, 2rj - 2r + 2, \dots, 2rj\},$$

i.e. $\mathbb{A}_j$ is the set of natural numbers that are "moved" by A_j. We also set

$$J := \{j \mid A_j B \neq B A_j\}. \tag{14}$$

If $j \notin J$ then d_j is relatively prime to $2r$, because $A_j^{d_j}$ is a cycle of length $2r$. Clearly, if the type of $\Phi(B)$ is (L) then $\operatorname{card} J = L$. So, we can write

$$\gamma_{l1} = \prod_{j \notin J} A_j^{d_j} \quad \prod_{j \in J} A_j^{d_j} B, \tag{15}$$

and it is easy to see that g_l can be chosen

$$g_l = \nu \prod_{j \notin J} \sigma_{l,j} \ , \tag{16}$$

where $\sigma_{l,j} A_j \sigma_{l,j}^{-1} = A_j^{d_j}$ and every element of $\mathbb{A}_{j'}$ is fixed by $\sigma_{l,j}$ if $j \notin J$ and $j' \neq j$, and ν is such that

$$\nu \prod_{j \in J} A_j \ \nu^{-1} = \prod_{j \in J} A_j^{d_j} B, \tag{17}$$

and that every element of $\mathbb{A}_j$, $j \notin J$, is fixed by ν.

Lemma 2.17. *$\Phi(B)$ and $\Phi(B')$ have the same type in $\mathbb{S}_n$.*

Proof. We will consider cases according to the type of $\Phi(B)$ in $\mathbb{S}_n$.

If the type of $\Phi(B)$ is (1^{2rn}); this means $B = \mathrm{id}$. We have that $J = \emptyset$, so we can chose $g_l = \sigma_{l,1} \cdots \sigma_{l,n}$. Then

$$\gamma_{1l} = g_l^{-1} \pi g_l = \sigma_{l,1}^{-1} A_1 \sigma_{l,1} \cdots \sigma_{l,n}^{-1} A_n \sigma_{l,n},$$

and since γ_{1l} is in $\mathcal{C}$, i.e. it is a product of disjoint cycles of length $2r$, we have that $\sigma_{l,j}^{-1} A_j \sigma_{l,j}$ is a cycle of length $2r$, for all j. This implies that

$$\gamma_{1l} = A_1^{e_1} \cdots A_n^{e_n},$$

with $e_1, \dots, e_n$ relatively primes to $2r$; this means that $B' = \mathrm{id}$.

If the type of $\Phi(B)$ is (2). It is enough to assume that $B = B_i$ for some i, $1 \le i \le n-1$. We saw that if $j \neq i, i+1$ then d_j is relatively prime to $2r$, and that g_l can be chosen as in (16), i.e.

$$g_l = \nu \prod_{j \neq i, i+1} \sigma_{l,j} \ ,$$

where ν satisfies $\nu A_i A_{i+1} \nu^{-1} = A_i^{d_i} A_{i+1}^{d_{i+1}} B$, and if $j \neq i, i+1$ then the elements of $\mathbb{A}_j$ are fixed by ν. Hence,

$$\gamma_{1l} = g_l^{-1} \pi g_l = \prod_{j \neq i, i+1} \sigma_{l,j}^{-1} A_j \sigma_{l,j} \quad \nu^{-1} A_i A_{i+1} \nu = \prod_{j \neq i, i+1} A_j^{e_j} \quad A_i^{e_i} A_{i+1}^{e_{i+1}} B',$$

with e_j relatively prime to $2r$, if $j \neq i, i+1$. This implies that the type of $\Phi(B')$ is $(h_1^{m_1}, \dots, h_K^{m_K})$ with

$$m_1 h_1 + \dots + m_K h_K \leq 2.$$

Then the type of $\Phi(B')$ is (1) or (2); if it is (1) we have that $B' = \mathrm{id}$, then $B = \mathrm{id}$, by the first case, a contradiction. Thus, the type of $\Phi(B')$ is (2).

Notice that if the type of $\Phi(B)$ is (2^a) then the same occurs for $\Phi(B')$, by repeating the previous argument in each disjoint transposition that appears in the decomposition of $\Phi(B)$ as product of disjoint permutations of $\mathbb{S}_n$.

In general, we can prove by the same argument that if the result is true when the type of $\Phi(B)$ is (L_1) and (L_2) then the result is also true if the type of $\Phi(B)$ is (L_1^2), if $L_1 = L_2$, or (L_1, L_2), if $L_1 \neq L_2$.

Let $\Phi(B)$ be of type (L). We use induction in L and the previous paragraphs to prove that the type of $\Phi(B')$ is (L). Explicitly, assume that there exists $L > 2$ such that for every $h < L$ it is true the following: if the type of $\Phi(B)$ is (h), then the type of $\Phi(B')$ is (h). Suppose that the type of $\Phi(B')$ is $(h_1^{m_1}, \dots, h_K^{m_K})$. We proceed as in the case $L = 2$. We can chose g_l as in (16), with ν that satisfies (17), and if $j \notin J$ then the elements of $\mathbb{A}_j$ are fixed by ν. Hence

$$\gamma_{1l} = g_l^{-1} \pi g_l = \prod_{j \notin J} \sigma_{l,j}^{-1} A_j \sigma_{l,j} \quad \nu^{-1} \prod_{j \in J} A_j \, \nu = \prod_{j \notin J} A_j^{e_j} \quad \prod_{j \in J} A_j^{e_j} B',$$

with e_j relatively prime to $2r$ if $j \notin J$, because γ_{1l} is in $\mathcal{C}$. This implies that

$$m_1 h_1 + \dots + m_K h_K \leq L.$$

If $m_1 h_1 + \dots + m_K h_K < L$ or if $m_1 h_1 + \dots + m_K h_K = L$ with $K > 1$, then $h_1, \dots, h_K < L$, and by inductive hypothesis and the previous paragraph we have that the type of $\Phi(B)$ is $(h_1^{m_1}, \dots, h_K^{m_K}) \neq (L)$, which is a contradiction. So, type of $\Phi(B')$ is $(h_1)^{m_1}$, with $m_1 h_1 = L$; if $m_1 > 1$ we use inductive hypothesis and the previous paragraph to say that the type of $\Phi(B)$ is $(h_1^{m_1}) \neq (L)$, which is a contradiction. Hence, $m_1 = 1$ and $h_1 = L$, it means that the type of $\Phi(B') = (L)$, and this concludes the proof. □

Lemma 2.18. *Let γ_{l1} and γ_{1l} as in* (12) *and* (13)*, respectively.*
(a) For any r if n is odd, then $\sum_{j=1}^{n}(e_j + d_j)$ is even.
(b) If r is even and n is even, then $\sum_{j=1}^{n}(e_j + d_j) \equiv 0 \mod (4)$.

Proof. (a) If n is odd we have that the sign of π in $\mathbb{S}_{2rn}$ is

$$\operatorname{sgn} \pi = \operatorname{sgn} A_1 \cdots \operatorname{sgn} A_n = (-1)^n = -1,$$

because $A_1, \dots, A_n$ are cycles of even length. Since $\gamma_{l1} \in \mathcal{C}$ we have that $\operatorname{sgn} \gamma_{l1} = -1$, on the other hand

$$\operatorname{sgn} \gamma_{l1} = \operatorname{sgn} A_1^{d_1} \cdots \operatorname{sgn} A_n^{d_n} \operatorname{sgn} B = (-1)^{d_1+\cdots+d_n},$$

because $B \in \langle B_1, \dots, B_{n-1} \rangle$ and every $B_1, \dots, B_{n-1}$ is a product of an even number of transpositions in $\mathbb{S}_{2rn}$. Then $d_1 + \cdots + d_n$ is odd. Analogously, $e_1 + \cdots + e_n$ is odd. Then the result follows.

(b) Assume that n is even. In this case the sign of π in $\mathbb{S}_{2rn}$ is 1; since γ_{l1} and γ_{1l} are in $\mathcal{C}$, $d_1 + \cdots + d_n$ and $e_1 + \cdots + e_n$ are even. We suppose that the decomposition of $\Phi(B)$ in product of disjoint permutation in $\mathbb{S}_n$ is

$$\Phi(B) = \tau_1 \cdots \tau_K. \tag{18}$$

By Lemma 2.17, we have that

$$\Phi(B') = \tau_1' \cdots \tau_K',$$

with $|\tau_k| = |\tau_k'|$, for all k. Obviously, $|B| = \operatorname{lcm}(|\tau_1|, \dots, |\tau_K|) = |B'|$.

For every k, $1 \le k \le K$, we define

$$J_k := \{ j \mid 1 \le j \le n \text{ and } A_j \Phi^{-1}(\tau_k) \neq \Phi^{-1}(\tau_k) A_j \}. \tag{19}$$

Clearly, $\operatorname{card} J_k = |\tau_k|$, for all k. Note that $J_1, \dots, J_K$ are disjoint sets and if J is as in (14) then $J = J_1 \cup \cdots \cup J_K$. Besides, it is clear that

$$J_k = \{ j \mid 1 \le j \le n \text{ and } A_j \Phi^{-1}(\tau_k') \neq \Phi^{-1}(\tau_k') A_j \}.$$

by Lemma 2.17. We write γ_{l1} as in (15) in a more precise form

$$\gamma_{l1} = g_l \pi g_l^{-1} = \prod_{j \notin J} A_j^{d_j} \quad \prod_{j \in J_1} A_j^{d_j} \cdots \prod_{j \in J_K} A_j^{d_j} \quad B,$$

and g_l can be chosen as in (16)

$$g_l = \nu_1 \cdots \nu_K \prod_{j \notin J} \sigma_{l,j} \ ,$$

where

$$\nu_k \prod_{j \in J_k} A_j \ \nu_k^{-1} = \prod_{j \in J_k} A_j^{d_j} \ \Phi^{-1}(\tau_k), \tag{20}$$

and if $j \notin J_k$ every element of $\mathbb{A}_j$ is fixed by ν_k; this allows to say that if $j \notin J_k$ then A_j and ν_k commute. Hence, if γ_{1l} is as in (13) then

$$\gamma_{1l} = g_l \pi g_l^{-1} = \prod_{j \notin J} A_j^{e_j} \quad \prod_{j \in J_1} A_j^{e_j} \cdots \prod_{j \in J_K} A_j^{e_j} \quad B',$$

with

$$\prod_{j\in J_k} A_j^{e_j}\Phi^{-1}(\tau'_k) = \nu_k^{-1}\prod_{j\in J_k} A_j\ \nu_k. \tag{21}$$

Since $|\gamma_{l1}| = 2r$, then $B^{2r} = \mathrm{id}$; this implies that $|B|$ divides $2r$, let us say $2r = |B|q$, with $q \geq 1$. It is straightforward to prove that

(22)

$$\left(\prod_{j\in J} A_j^{d_j} B\right)^{h|B|} = \left(\prod_{j\in J_1} A_j\right)^{h\frac{|B|}{|\tau_1|}\sum_{j\in J_1} d_j} \cdots \left(\prod_{j\in J_K} A_j\right)^{h\frac{|B|}{|\tau_K|}\sum_{j\in J_K} d_j},$$

for all integer h. When $h = q$ both sides are equal to id and this implies

$$\left(\prod_{j\in J_k} A_j\right)^{q\frac{|B|}{|\tau_k|}\sum_{j\in J_k} d_j} = \mathrm{id},$$

for all k. Since the order of $\prod_{j\in J_k} A_j$ is $2r$ we have that $|\tau_k|$ divides $\sum_{j\in J_k} d_j$. Analogously, we can prove that $|\tau'_k|$ divides $\sum_{j\in J_k} e_j$, for all K. Hence, for every k, $1 \leq k \leq K$, there exist $p_k, p'_k \geq 1$ such that

$$\sum_{j\in J_k} d_j = |\tau_k|p_k \qquad \text{and} \qquad \sum_{j\in J_k} e_j = |\tau_k|p'_k. \tag{23}$$

By (20), (21) and (22), for every k we have that

$$\begin{aligned}
\left(\prod_{j\in J_k} A_j\right)^{h|B|} &= \nu_k^{-1}\nu_k\left(\prod_{j\in J_k} A_j\right)^{h|B|}\nu_k^{-1}\nu_k = \nu_k^{-1}\left(\nu_k\prod_{j\in J_k} A_j\nu_k^{-1}\right)^{h|B|}\nu_k \\
&= \nu_k^{-1}\left(\prod_{j\in J_k} A_j^{d_j}\Phi^{-1}(\tau_k)\right)^{h|B|}\nu_k = \nu_k^{-1}\left(\prod_{j\in J_k} A_j\right)^{h|B|p_k}\nu_k \\
&= \left(\nu_k^{-1}\prod_{j\in J_k} A_j\nu_k\right)^{h|B|p_k} = \left(\prod_{j\in J_k} A_j^{e_j}\Phi^{-1}(\tau'_k)\right)^{h|B|p_k} \\
&= \left(\prod_{j\in J_k} A_j\right)^{h|B|p_kp'_k},
\end{aligned}$$

for all integer h. In particular, for $h = 1$ this implies that $2r$ divides $|B|p_kp'_k - |B|$. Since $2r = |B|q$, we have that q divides $p_kp'_k - 1$, for every k; let us say that for every k, $1 \leq k \leq K$, there exists $x_k \geq 1$ such that

$$p_kp'_k - 1 = qx_k. \tag{24}$$

By a similar argument as in the previous paragraph, we can show that

$$\left(\prod_{j\in J_k} A_j\right)^{h|\tau_k|} = \left(\prod_{j\in J_k} A_j\right)^{h|\tau_k|p_k p'_k}, \tag{25}$$

for all integer h. For $h = 1$, this says that $2r$ divides $|\tau_k|p_k p'_k - |\tau_k|$. Using (24) and that $|B| = |\tau_k|y_k$, for some $y_k \geq 1$, we have that y_k divides x_k, for every k, let us say $x_k = y_k z_k$, for some $z_k \geq 1$. Hence

$$|\tau_k|p_k p'_k - |\tau_k| = |\tau_k|\, q\, y_k\, z_k = |B|\, q\, z_k = 2r z_k.$$

Since r is even we have that $|\tau_k|p_k p'_k \equiv |\tau_k| \mod (4)$; this means that

$$p'_k \sum_{j\in J_k} d_j \equiv |\tau_k| \mod (4) \quad \text{and} \quad p_k \sum_{j\in J_k} e_j \equiv |\tau_k| \mod (4). \tag{26}$$

Clearly,

$$(\sum_{j\in J_k} d_j)\, (\sum_{j\in J_k} e_j) \equiv |\tau_k|^2 \mod (4). \tag{27}$$

Using (26), (27) and that $|\tau_k|^2 \equiv 0$ or $1 \mod (4)$ we conclude that

$$\sum_{j\in J_k} d_j \equiv \sum_{j\in J_k} e_j \mod (4), \tag{28}$$

for every k, $1 \leq k \leq K$. Moreover, if $|\tau_k|^2 \equiv 0 \mod (4)$, then

$$\sum_{j\in J_k} d_j \equiv 0 \equiv \sum_{j\in J_k} e_j \mod (4) \quad \text{or} \quad \sum_{j\in J_k} d_j \equiv 2 \equiv \sum_{j\in J_k} e_j \mod (4), \tag{29}$$

and if $|\tau_k|^2 \equiv 1 \mod (4)$, then

$$\sum_{j\in J_k} d_j \equiv 1 \equiv \sum_{j\in J_k} e_j \mod (4) \quad \text{or} \quad \sum_{j\in J_k} d_j \equiv 3 \equiv \sum_{j\in J_k} e_j \mod (4). \tag{30}$$

For $h = 0$, 1, 2 and 3, we define

$$\mathcal{K}_h := \{k \,|\, 1 \leq k \leq K \text{ and } |\tau_k|^2 \equiv h \mod (4)\}.$$

Then we can write

$$\sum_{j=1}^{n}(e_j + d_j) = \sum_{j\notin J}(e_j + d_j) + \sum_{k\in\mathcal{K}_1\cup\mathcal{K}_3}\sum_{j\in J_k}(e_j + d_j) + \sum_{k\in\mathcal{K}_0\cup\mathcal{K}_2}\sum_{j\in J_k}(e_j + d_j).$$

By (29), it is clear that

$$\sum_{k\in\mathcal{K}_0\cup\mathcal{K}_2}\sum_{j\in J_k}(e_j + d_j) \equiv 0 \mod (4),$$

while if $k \in K_1 \cup \mathcal{K}_3$ then $\sum_{j\in J_k}(e_j + d_j) \equiv 2 \mod (4)$. Besides, if $j \notin J$ then d_j and e_j are relatively prime to $2r$ and it is easy to see that

$$A_j = \sigma_{l,j}^{-1}\sigma_{l,j}A_j\sigma_{l,j}^{-1}\sigma_{l,j} = \sigma_{l,j}^{-1}A_j^{d_j}\sigma_{l,j} = (\sigma_{l,j}^{-1}A_j\sigma_{l,j})^{d_j} = A_j^{e_j d_j}; \tag{31}$$

this implies that $2r$ divides $e_jd_j - 1$, and since r is even we have $e_jd_j \equiv 1 \mod (4)$. Using that d_j and e_j are odd and the last fact we can prove that

$$e_j + d_j \equiv 2 \mod (4), \tag{32}$$

for every $j \notin J$.

We saw that $\operatorname{card} J_k = |\tau_k|$, hence $\sum_{k\in K_0\cup\mathcal{K}_2} \operatorname{card} J_k$ is even. Since

$$n = \operatorname{card} J^c + \sum_{k\in K_1\cup\mathcal{K}_3} \operatorname{card} J_k + \sum_{k\in K_0\cup\mathcal{K}_2} \operatorname{card} J_k$$

is even we have that $a := \operatorname{card} J^c + \sum_{k\in K_1\cup\mathcal{K}_3} \operatorname{card} J_k$ is even. Hence

$$\sum_{j=1}^{n} (e_j + d_j) \equiv 2^a \equiv 0 \mod (4), \tag{33}$$

and the result follows. □

Proof of Theorem 2.16 Let t_h, t_l in $\mathcal{C}$ that commute; it amounts to say that $\gamma_{hl} := g_l^{-1} t_h g_l$ and $\gamma_{lh} := g_h^{-1} t_l g_h$ are in $\mathbb{S}^{\pi}_{2rn}$. Let $\rho \in \widehat{\mathbb{S}^{\pi}_{2rn}}$ as in the statements (a) or (b). Let us remember from subsection 1.2, that $q_{hh} = \rho(\gamma_{hh})$, $q_{ll} = \rho(\gamma_{ll})$, $q_{hl} = \rho(\gamma_{hl})$ and $q_{lh} = \rho(\gamma_{lh})$. We must see that $q_{hh} = -1 = q_{ll}$ and $q_{hl}q_{lh} = 1$. The first conditions are trivially fulfilled. For the last one we consider two cases.

CASE 1: $h = 1$. Let γ_{l1}, γ_{1l} be as in (12) and (13), respectively.

(i) *Assume that* $\rho = \chi_{r,\dots,r} \otimes \mu$, *with* $\mu = \epsilon$ *or* sgn. Since $\rho(\pi) = -1$ and $\rho(\pi) = \omega^{rn}$, with $\omega = \exp(\frac{i\pi}{r})$, we have that n must be odd. Then

$$\begin{aligned} q_{1l}q_{l1} &= (\chi_{r,\dots,r} \otimes \mu)(\gamma_{1l}\gamma_{l1}) = (\chi_{r,\dots,r} \otimes \mu)(\gamma_{1l})\ (\chi_{r,\dots,r} \otimes \mu)(\gamma_{l1}) \\ &= \omega^{r\sum_{j=1}^{n} e_j + d_j} \mu(B')\mu(B) = (-1)^{\sum_{j=1}^{n} e_j + d_j} = 1, \end{aligned}$$

by Lemma 2.17 and Lemma 2.18 (a).

(ii) *Assume that r is even and* $\rho = \chi_{c,\dots,c} \otimes \mu$, *with* $c = \frac{r}{2}$ *or* $\frac{3r}{2}$, *and* $\mu = \epsilon$ *or* sgn. The condition $\rho(\pi) = -1$ implies that $n \equiv 2 \mod (4)$; in particular n is even. By Lemma 2.17 and Lemma 2.18 (b), we can say

$$\begin{aligned} q_{1l}q_{l1} &= (\chi_{c,\dots,c} \otimes \mu)(\gamma_{1l}\gamma_{l1}) = (\chi_{c,\dots,c} \otimes \mu)(\gamma_{1l})\ (\chi_{c,\dots,c} \otimes \mu)(\gamma_{l1}) \\ &= \omega^{c\sum_{j=1}^{n} e_j + d_j} \mu(B')\mu(B) = (\pm i)^{\sum_{j=1}^{n} e_j + d_j} = 1. \end{aligned}$$

CASE 2: $h \neq 1$. We call $\tilde{\pi} := t_h$.

(i) *Assume that* $\rho = \chi_{r,\dots,r} \otimes \mu$, *with* $\mu = \epsilon$ *or* sgn. Then there exists $\tilde{c}$, with $0 \le \tilde{c} \le 2r - 1$, such that $\chi_{\tilde{c},\dots,\tilde{c}} \otimes \widetilde{\mu} \in \widehat{\mathbb{S}^{\tilde{\pi}}_{2rn}}$ and

$$M(\mathcal{C}, \chi_{r,\dots,r} \otimes \mu) = M(\mathcal{C}, \chi_{\tilde{c},\dots,\tilde{c}} \otimes \widetilde{\mu}), \tag{34}$$

where $\widetilde{\mu} = \epsilon$ or sgn, say $\widetilde{\mu} = $ sgn. This implies that $\widetilde{\rho} := \chi_{\tilde{c},\dots,\tilde{c}} \otimes \widetilde{\mu}$ and ρ have the same image, see (3), it means that $\langle \pm\omega^{\tilde{c}} \rangle = \langle \omega^r \rangle = \{1, -1\}$. Since $\omega = \exp(\frac{i\pi}{r})$ it is clear that $\tilde{c} = r$. Now, the result follows for $\widetilde{\rho}$ from the case (1)(i). The case $\widetilde{\mu} = \epsilon$ is similar.

(ii) *Assume that r is even and $\rho = \chi_{c,\dots,c} \otimes \mu$, with $c = \frac{r}{2}$ or $\frac{3r}{2}$, and $\mu = \epsilon$ or* sgn. Then there exists $\tilde{c}$, with $0 \leq \tilde{c} \leq 2r-1$, such that

$$M(\mathcal{C}, \chi_{c,\dots,c} \otimes \mu) = M(\mathcal{C}, \chi_{\tilde{c},\dots,\tilde{c}} \otimes \widetilde{\mu}), \tag{35}$$

where $\widetilde{\mu} = \epsilon$ or sgn, say $\widetilde{\mu} =$ sgn. This implies that $\widetilde{\rho} := \chi_{\tilde{c},\dots,\tilde{c}} \otimes \widetilde{\mu}$ and ρ have the same image, i.e. $\langle \pm\omega^{\tilde{c}} \rangle = \langle \omega^r \rangle = \{1, i, -1, -i\}$. Since $\omega = \exp(\frac{i\pi}{r})$ it is clear that $\tilde{c} = \frac{r}{2}$ or $\frac{3r}{2}$. Now, the result follows for $\widetilde{\rho}$ from the case (1)(ii). The case $\widetilde{\mu} = \epsilon$ is similar.

This concludes the proof. □

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